Percolations on Hypergraphs

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We offer analytical solutions to classical percolation problems on hypergraphs with arbitrary vertex degree and hyperedge cardinality distributions. We introduce a generalization of the 2-core for hypergraph and we show that it can emerge in either a continuous or a hybrid percolation transition. We also define two different hypergraph cores related to the hyperedge cover and vertex cover problems on hypergraphs. We validate our analytical results with extensive numerical simulations.

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I. INTRODUCTION

A hypergraph is a natural generalization of a graph, where an edge (often called hyperedges) can simultaneously connect any number of vertices [5]. As in graphs, where the degree of a vertex in a hypergraph is the number of hyperedges that connect to it. The number of vertices connected by a hyperedge is called the cardinality of that hyperedge. If all hyperedges have the same cardinality K, the hypergraph is said to be uniform or K-uniform. Note that a graph is just a 2-uniform hypergraph. The fact that hyperedges can connect more than two vertices facilitates a more precise representation of many real-world networks. For example, collaboration network can be typically represented by a hypergraph, where vertices represent individuals and hyperedges connect individuals who were involved in a specific collaboration, e.g., a scientific paper, a patent, a consulting task, or an art performance [17, 30]. Many cellular networks can also be represented by hypergraphs [16]. For example, given a set of proteins and a set of protein complexes, the corresponding hypergraph naturally captures the information on proteins that occurred together in a protein complex. For a biochemical reaction system, the hypergraph representation will indicate which bimoleculars participate in a particular reaction [16, 27]. In computer science, the factorization of complicated global functions of many variables can often be represented by a factor graph, a bipartite graph that manifest which variables are arguments of which local functions [18]. A factor graph is equivalent to a hypergraph, where the nodes represent the variables and the hyperedges represent the local functions.

Despite the ubiquity of hypergraphs in different fields, fundamental structural properties of hypergraphs have not been fully understood. Most of the previous works focus on uniform hypergraphs [7, 12, 29], ignoring the fact that hyperedges could have a wide range of cardinalities. In this work, we systematically study the percolation transitions on hypergraphs with arbitrary vertex degree and hyperedge cardinality distributions. We are particularly interested in the emergence of a giant component, the K-core, and the core in hypergraphs (see Fig. 1). Those special subgraphs have been extensively studied in the graph case and play very important roles in many network properties [1, 8]. A giant component of a graph is a connected component that contains a constant fraction of the entire graph's vertices, which is relevant to structural robustness and resilience of networks [9, 10]. The K-core of a graph is obtained by recursively removing vertices with degree less than K, as well as edges incident to them. The K-core has been used to identify influential spreaders in complex networks [15]. The core of a graph is the remainder of the greedy leaf removal (GLR) procedure: leaves (vertices of degree one) and their neighbors are removed iteratively from the graph. The emergence of the core in a graph has been related to the conductor-insulator transition [3], structural controllability [22], and many combinatorial optimization problems [14].

We can naturally extend the definition of giant component to the hypergraph case. Yet, to obtain the K-core in a hypergraph, we have to specify how to remove hyperedges containing vertices of degree less than K. To achieve that, we introduce the (K, S)-core defined as the largest fraction of the hypergraph where each hyperedge contains at least S nodes and each vertex belongs to at least K hyperedges in the subset. The (K, S)-core is obtained by recursively removing vertices with degree less than K and hyperedges with cardinality less than S. We can generalize the GLR procedure to the hypergraph case in two slightly different ways, rendering two different cores. Core1 is the ramainder hypergraph obtained by recursively removing all the hyperedges(nodes) with cardinaltiy(degree) less than two; all the hyperedges that contain those nodes, along side with all the vertices contained in them. Core2 is the ramainder hypergraph obtained by recursively removing all the hyperedges(nodes) with cardinaltiy(degree) less than two; all the nodes that contained on those hyperedges, as well as all the hyperedges that contain them.

In this work we offer analytical solutions to those classical percolation problems on hypergraphs with arbitrary vertex degree and hyperedge cardinality distribu-

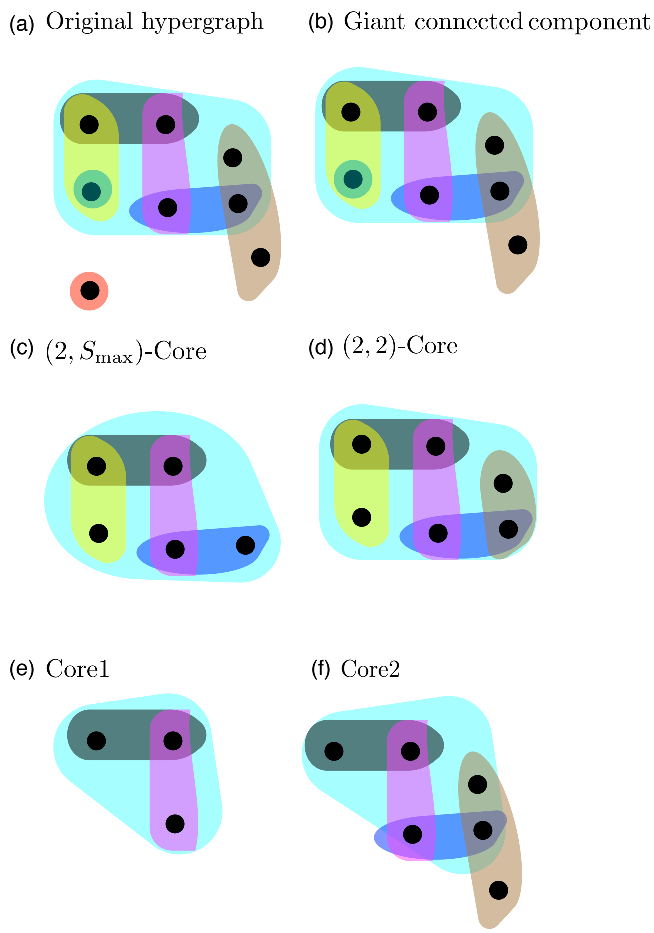

FIG. 1. (Color online) Example of the diferent percolations stuided in the work. (a) shows the original hypergraph. (b), (c) , (d), (e) and (f) shows the $(2, S_{\max})$-core, $(2,2)$-core, core1 and core2 of the same hypergraph. Here we choose we $S_{\max} \equiv \max(r, 2)$.

tions. We confirm all our results using extensive numerical simulations (SI Sec.III). Interestingly, we find that the above-mentioned core1 and core2 are related to the minimum edge cover and vertex cover problems, respectively. Both problems are classical combinatorial optimization problems and have a wide range of applications from the detection of potential data races in multithreaded programs to drug selection for cancer therapy [26**?**]. In the hyperedge (or vertex) cover problem we aim to find the minimum set of hyperedges (or vertices) that cover all the vertices (or hyperedges) in the hypergraph. Note that in the graph case, the edge cover problem can be solved in polynomial time [20]. While for general hypergraphs, the hyperedge cover problem can be NP-complete [19]. The vertex cover problem is generally NP-complete for both graphs and hypergraphs [2, 20].

II. GIANT COMPONENT

A giant component of a hypergraph is a connected component that contains a constant fraction of the entire vertices. In the mean-field picture, we can derive a set of self-consistent equations to calculate the relative size of the giant component, using the generating function formalism [25]. Let μ represent the probability that a randomly selected vertex from a randomly chosen hyperedge is not connected via other hyperedges with the giant component. Dually, let ψ represent the probability that a randomly chosen hyperedge connecting to a randomly chosen vertex is not connected via other vertices with the giant component. Then we have

$$\mu = \sum_{k=1}^{\infty} Q_{\rm n}(k)\psi^{k-1} \tag{1}$$

$$\psi = \sum_{r=1}^{\infty} Q_{\rm h}(r)\mu^{r-1}. \tag{2}$$

Here $Q_{\rm n}(k) \equiv kP_{\rm n}(k)/c$ is the excess degree distribution of vertices, i.e., the degree distribution for the vertices in a randomly chosen hyperedge. $P_{\rm n}(k)$ is the vertex degree distribution, and $c = c_1$ is the mean degree of the vertices. In general we define $c_m \equiv \sum_{k=0}^{\infty} k^m P_{\rm n}(k)$. $Q_{\rm h}(r) \equiv rP_{\rm h}(r)/d$ is the excess cardinality distribution of hyperedges, i.e., the cardinality distribution for the hyperedges connected to a randomly chosen vertex. $P_{\rm h}(r)$ is the hyperedge cardinality distribution, and $d = d_1$ is the mean cardinality of the hyperdges. In general we define $d_m \equiv \sum_{r=0}^{\infty} r^m P_{\rm h}(r)$.

The relative size of the giant component is then given by

$$s_{\rm g} = 1 - \sum_{k=0}^{\infty} P_{\rm n}(k)\psi^k. \tag{3}$$

Fig. 2 shows the analytical result of $s_{\rm g}$ as a function of the mean degree c for hypergraphs with Poisson vertex degree distribution and different hyperedge cardinality distributions. Clearly the giant component in hypergraphs emerges as a continuous phase transition with scaling behavior

$$s_{\rm g} \sim (c - c^*)^{\eta} \tag{4}$$

for $c - c^* \to 0^+$, where c^* is the critical value of mean degree (i.e., the percolation threshold) and η is the critical exponent associated with the critical singularity.

The condition for the percolation transition can be determined by differentiating both sides of Eq. (1) over μ and then evaluating at $\mu = 1$, yielding

$$\frac{d_2 - d}{d}\frac{c_2 - c}{c} > 1. \tag{5}$$

(See SI Sec.I for details.) Note that a similar relation has been found for uniform hypergraphs [23]. In the graph case ($d=2$ for all edges) we recover the classical result $\frac{c_2-c}{c}>1$ [9, 24].

The critical exponent η can be calculated by expanding Eq. (3) in powers of $(c-c^*)$ around the critical point c^* (see SI Sec.II). For hypergraphs with bounded moments of cardinality distribution and degrees distribution, we obtain the same exponent $\eta=1$ as in the graph case[4, 9].

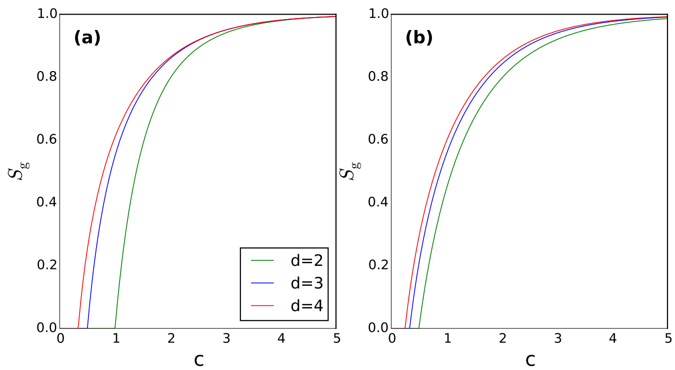

FIG. 2. (Color online) The relative size of the giant component $s_{\rm g}$ as a function of the mean degree c for hypergraphs with Poisson vertex degree distribution. (a) d-uniform hypergraphs with $d=2,3,4$; (b) hypergraphs with Poisson hyperedge cardinality distribution and mean cardinality $d=2,3,4$.

III. (K,S)-CORE

The (K,S)-core of a hypergraph is obtained by recursively removing vertices with degree less than K and hyperedges with cardinality less than S. A hyperedges with cardinality r is removable if at least $r-S+1$ vertices connected to it are also removable and a vertex with degree k is removable if at least $k-K+1$ hyperedges connected to it are also removable. One can remove a vertex or a hyperedge from the hypergraph, and see what is the probability of a neighboring hyperedge or vertex, respectively, being removable. This allows us to derive a set of self-consistent equations:

$$\alpha=\sum_{k=1}^{\infty}Q_{\rm n}(k)\sum_{l=k+1-S}^{k-1}\binom{k-1}{l}\delta^l(1-\delta)^{k-1-l}, \quad (6)$$

$$\delta=\sum_{r=1}^{\infty}Q_{\rm h}(r)\sum_{l=r+1-K}^{r-1}\binom{r-1}{l}\alpha^l(1-\alpha)^{r-1-l}. \quad (7)$$

where α and δ are, respectively, the probability that a vertex or a hyperedge is removable. From now on we will focus on the case of $K=2$. Then Eq. (6) reduces to

$$\alpha=\sum_{k=1}^{\infty}Q_{\rm n}(k)\delta^{k-1}. \quad (8)$$

A. $K=2$ and $S=S_{\max}$

The (K,S)-core defined with $K=2$ and $S=S_{\max}\equiv\max(r,2)$ is obtained by recursively removing all vertices with degree one as well as the hyperedges containing them, and all hyperedges with cardinality smaller than two. Hyperedges with cardinality one or zero do not connect any nodes, thus have no meaning in what cores are concerned. In this case the threshold S depends on the cardinally of the hyperedges. Furthermore, if one of the vertices of any hyperedge is removed the hyperedge is also removed. (Note that the $(2,S_{\max})$-core has been defined in literature [7] simply as 2-core, and discontinuous 2-core percolation is found in d-uniform hypergraphs with $d>2$.) In this case, Eq. (7) reduces to

$$1-\delta=\sum_{r=2}^{\infty}Q_{\rm h}(r)(1-\alpha)^{r-1}. \quad (9)$$

The relative size of the $(2,S_{\max})$-core is given by the probability that a randomly chosen vertex is connected to at least two non-removable hyperedges:

$$s_{2\rm c}=\sum_{k=2}^{\infty}P_{\rm h}(k)\sum_{l=2}^{k}\binom{k}{l}(1-\delta)^l\delta^{k-l}. \quad (10)$$

Fig. 3 shows the analytical result of $s_{2\rm c}$ as a function of the mean degree c for hypergraphs with Poisson vertex degree distribution and different hyperedge cardinality distributions. We find that, depending on the mean hyperedge cardinality d, the $(2,S_{\max})$-core emerges as either a continuous or a hybrid phase transition, with scaling behavior

$$s_{2\rm c}-s^*_{2\rm c}\sim(c-c^*)^{\zeta} \quad (11)$$

for $c-c^*\to 0^+$, where c^* is the percolation threshold and ζ is the critical exponent. $s^*_{2\rm c}$ is the $(2,S_{\max})$-core relative size right at the critical point: $s^*_{2\rm c}=0$ for continuous phase transitions and non-zero for hybrid phase transitions. The percolation threshold c^* can be calculated by differentiating both sides of Eq. (8) over α and then evaluating at the critical point, yielding

$$1=\sum_{k=1}^{\infty}\sum_{r=2}^{\infty}Q^*_{\rm n}(k)Q_{\rm h}(r)(k-1)(r-1){\delta^*}^{k-2}(1-\alpha^*)^{r-2}. \quad (12)$$

If the phase transition is continuous, this equation reduces to

$$Q_{\rm h}(2)\frac{c^*_2-c^*}{c^*}=1, \quad (13)$$

where c^*_2 is the second moment at the critical point. The phase transition is continuous if

$$2Q_{\rm h}(3)(c^*_2-c^*)-Q_{\rm h}(2)^2(c^*_3-3c^*_2+2c^*)<0. \quad (14)$$

(See SI Sec.I for details.) For d-uniform hypergraphs the $(2,S_{\max})$-core percolation is (i) continuous with critical

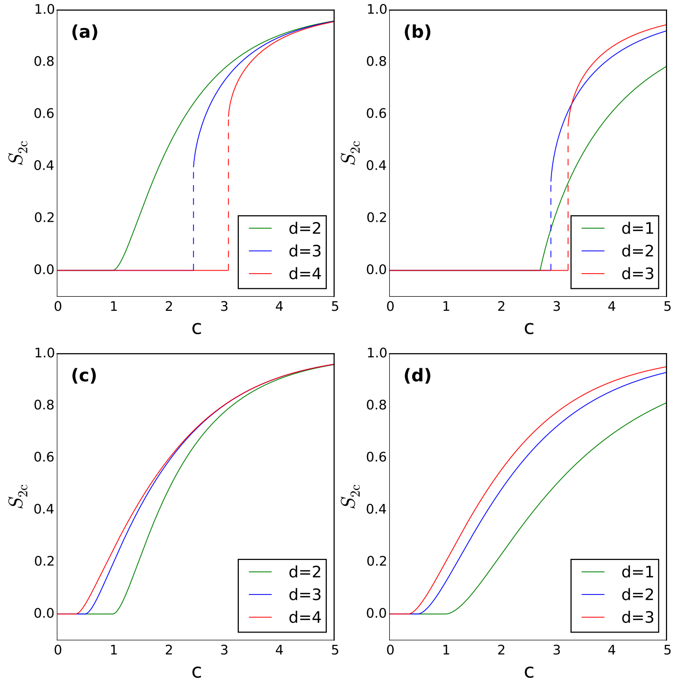

FIG. 3. (Color online) The relative size of $(2,S)$-core $s_{2\mathrm{c}}$ as a function of the mean degree c for hypergraphs with Poisson degree distribution. for $S = \max(r-1,2)$ (a) and (b), and $S = 2$, (b) and (d). (a) and (c) d-uniform hypergraphs . (b) and (d) hypergraphs with Poisson hyperedge cardinality distribution.

exponent $\zeta = 2$ if $d = 2$; and (ii) hybrid with critical exponent $\zeta = 1/2$ if $d > 2$ (which is consistent with a previous work [7]). For hypergraphs where both the vertex degeree and hyperedge cardnality distributions are Poissonian, the $(2,\max(r,2))$-core percolation is (i) continuous with critical exponent $\zeta = 2$ if $d < \bar{d} = 1$; (ii) continous with critical exponent $\zeta = 1$ if $d = \bar{d}$; and (iii) hybrid with critical exponent $\zeta = 1/2$ if $d > \bar{d}$. The same set of critical exponents was found for the heterogeneous-K-core [6].

B. $K = 2$ ans $S = 2$

In this section we study the $(2,2)$-core. A similar definition of removable hyperedges was used in [29], where the core obtained from the GLR procedure is used to study the vertex cover problem in uniform hypergraphs (See SI sec.II). In this case, Eq. (7) reduces to

$$\delta = \sum_{r=1}^{\infty} Q_{\mathrm{h}}(r)\alpha^{r-1}. \tag{15}$$

The relative size of the $(2,2)$-core can be calculated by considering the probability that a randomly chosen vertex is connected to at least two non-removable hyperedges and the probability that a degree-one vertex is

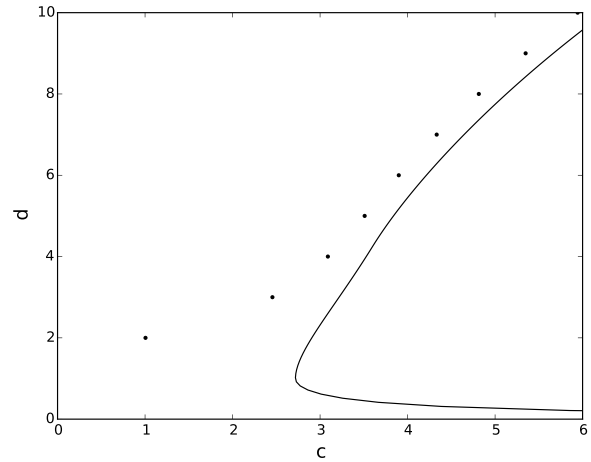

FIG. 4. Phase diagram of $(2,\max(r,2))$-core percolation on hypergraphs with Poisson vertex degree distributions. Black circles and black line represent the phase boundary of d-uniform hypergraphs and hypergraphs with Poisson hyperedge cardinality distribution, respectivly.

connected to a hyperedge with less than $(r-2)$ other degree-one vertices. This results in

$$s_{2\mathrm{c}} = \sum_{k=2}^{\infty} P_{\mathrm{n}}(k) \sum_{l=2}^{k} \binom{k}{l} (1-\delta)^{l} \delta^{k-l}. \tag{16}$$

Eqs. (10) and (15) have the same critical point as Eqs. (8) and (7). Therefore, for $(2,2)$-core we recover the result found in the graph case that both the $(2,2)$-core and the giant component emerge at the same critical point [9]. In this case the phase transition is always continuous (see solid lines in Fig. 3 c and d) and for the studied hyperedge cardinality and vertex degree distributions we have $\eta = 2$. The condition of percolation transition is still given by Eq. (5).

IV. THE CORE

A. core1

It is well known that the minimum edge cover problem on graphs can be computed in polynomial time [11]. Yet, this is not true for general hypergraphs [28]. The computational complexity of the minimum hyperedge cover problem can be related with the following GLR procedure. First, we remove all hyperedges with cardinaltiy one and hyperedges that contain vertices with degree one (called leaves) together with all the vertices contained in those hyperedges. Note that some of the vertices in those hyperedges may have degree larger than one. After we remove those vertices, the cardinality of other hyperedges that contain those vertices will decrease, and if the hyperedge cardinality drops below two, we remove those hyperedges. If eventually there is no core left, the difference between approximate solution from this GLR procedure and the exact solution is zero in the thermodynamic limit (see SI Sec.II for details).

To study the core percolation on hypergraphs, we generalize the mean-field approach proposed for the graph case [21]. We define two types of removable vertices: a vertex is (i) α-removable if it is or can become a vertex of degree one; (ii) β-removable if its degree is larger than one and belongs to at least one leaf hyperedge. Dually, we define two types of removable hyperedges: a hyperedge is (i) δ-removable if it is or can become an leaf hyperedge; (ii) ϵ-removable if it has cardinality r and is removed because it is connected to $(r-1)$ β-removable vertices. Consider a large uncorrelated random hypergraph $\mathcal{H}$ with arbitrary vertex degree and hyperedge cardinality distributions. We can determine the category of a vertex v in $\mathcal{H}$ by the categories of its neighboring hyperedges in the modified hypergraph $\mathcal{H}\setminus v$ with vertex v and all its hyperedges removed from $\mathcal{H}$, using the following rules: (i) α-removable vertex: all neighboring hyperedges are ϵ-removable; (ii) β-removable vertex: at least one neighboring hyperedge is δ-removable. Similarly, we can determine the category of a hyperedge e in $\mathcal{H}$ by the categories of its neighboring vertices in the modified hypergraph $\mathcal{H}\setminus e$ with hyperedge e and all its vertices removed from $\mathcal{H}$, using the following rules: (iii) δ-removable hyperedge: at least one neighboring vertex is α-removable; (iv) ϵ-removable hyperedge: at least one neighboring vertex is β-removable. Let α (or β) denote the probability that a random neighboring vertex of a random hyperedge e in a hypergraph $\mathcal{H}$ is α-removable (or β-removable) in $\mathcal{H}\setminus e$. Let δ (or ϵ) denote the probability that a random neighbor of a random vertex v in a hypergraph $\mathcal{H}$ is α-removable (or β-removable) in $\mathcal{H}\setminus v$. Then rules (i)-(iv) enable us to derive a set of self-consistent equations:

$$\alpha = \sum_{k=1}^{\infty} Q_{\rm n}(k)\epsilon^{k-1}, \tag{17}$$

$$1-\beta = \sum_{k=1}^{\infty} Q_{\rm n}(k)(1-\delta)^{k-1}, \tag{18}$$

$$1-\delta = \sum_{r=1}^{\infty} Q_{h}(r)(1-\alpha)^{r-1}, \tag{19}$$

$$\epsilon = \sum_{r=1}^{\infty} Q_{h}(r)\beta^{r-1}. \tag{20}$$

The relative size of core1 is given by

$$s_{\rm core1} = \sum_{k=2}^{\infty} P_n(k) \sum_{l=2}^{k} \binom{k}{l} (1-\delta-\epsilon)^{l}\epsilon^{k-l}. \tag{21}$$

For hypergraphs with Poisson vertex degree distribution and different hyperedge cardinality distributions, we find that the core emerges as a continuous phase transition (see Fig. 5),

$$s_{\rm core1} \propto (c-c^{*})^{\zeta} \tag{22}$$

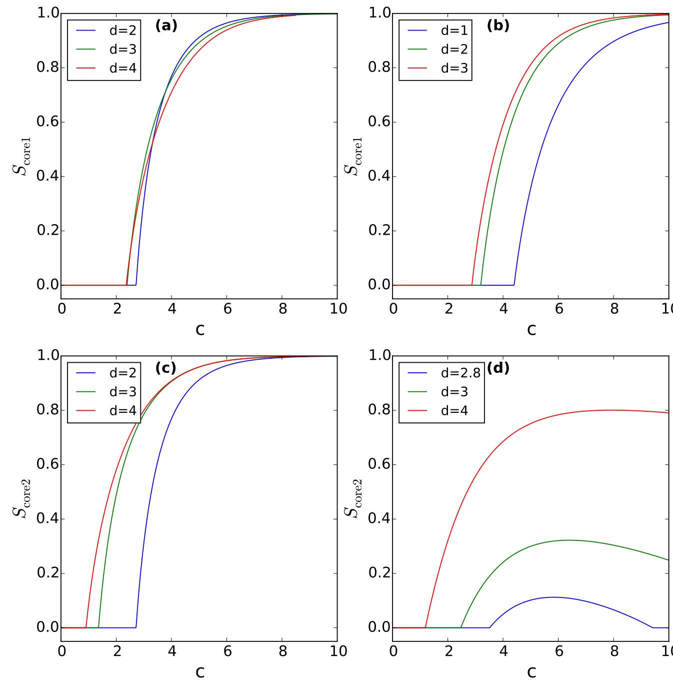

FIG. 5. (Color online) The relative core size of $s_{\rm core1}$ and $s_{\rm core1}$ for hypergraphs with Poisson degree distributions. (a) s_{c1} for d-uniform hypergraphs with $d = 2, 3.4$. s_{c1}, (b) and s_{c2},(d), for hypergraphs with Poisson hyperedge cardinality distribution and d-uniform hypergraphs for different values of d.

with critical exponent $\zeta = 1$ (see SI Sec.II for details). The relation between the critical mean degree c^{*} (percolation threshold) and the hyperedge mean cardinality d is represented in Fig. 6.

B. core2

We can define core2 that is associated with the vertex cover problems by considering the following procedure. First we remove all the vertices with degree one and all the vertices connected to hyperedges with cardinality one together with all the hyperedges connected to it. Note that some of the hyperedges connected to those vertices may have cardinality larger than one. After we remove those hyperedges (defined as leaves), the degree of the other vertices connected those hyperedges will decrease, and if the vertex degree drops below two, we remove those vertices. If the core size is non-zero, the vertex cover problem is, in general, an NP-hard problem [13].

There are two types of removable vertices: (i) α-removable vertices are vertices of degree one; (ii) β-removable vertices are vertices with degree larger than one that belong to at least one leaf hyperedge. Similarly, there are two types of removable hyperedges: (i) δ-removable hyperedges are hyperedges that are or can become leaf hyperedges; (ii) ϵ-removable hyperedges are hyperedges that are removed because they are connected to at least one β-removable vertex. The vertices obey the same relations as before, i.e., Eqs. (17) and (18), and for the hyperedges we derive the following self-consistant equations:

$$\delta = \sum_{r=1}^{\infty} Q_{\rm h}(r)\alpha^{r-1}, \tag{23}$$

$$1-\epsilon = \sum_{r=1}^{\infty} Q_{\rm h}(r)(1-\beta)^{r-1}. \tag{24}$$

Another way to obtain the same result is to use the fact that the core2 of a hypergraph is the core1 of the dual hypergraph (the dual of an hypergraph is a hypergraph whose vertices and hyperedges are interchanged). Thus, we can obtain the same results staring as shown in Eqs. eqs (17) to (20) by the following transformation,

$$\begin{aligned} \alpha &\to \delta, \\ \delta &\to \beta, \\ \beta &\to \epsilon, \\ \epsilon &\to \alpha, \\ Q_{\rm h}(r) &\rightleftarrows Q_{\rm n}(r). \end{aligned}$$

The relative size of the core $s_{\rm c}$ is given by the number of vertices connected to at least two non-removable hyperedges. Hence, we obtain

$$s_{\rm core2} = \sum_{k=2}^{\infty} P_{\rm n}(k) \sum_{l=2}^{k} \binom{k}{l} (1-\delta-\epsilon)^l \epsilon^{k-l}, \tag{25}$$

For hypergraphs with Poisson vertex degree distribution and different hyperedge cardinality distributions, we find that the core emerges as a continuous phase transition (see Fig. 5)

$$s_{\rm core2} \sim (c-c^*)^{\zeta_2}, \tag{26}$$

with critical exponent $\zeta = 1$ (see SI Sec.II for details). Fig. 5 (d) shows that for a Poisson-Poisson hypergraph the size of core2 starts to decrease at large values of c. By increasing the number of hyperlinks connected to a node, but keeping the cardinailty distribution constant, the probability of a node being connected to a hyperedge with cardinality one increases, and any node connected to a hyperedge with cardinality one is automatically removed. This effect is not relevant if the probability that a node is connect to a hyperedge with cardinality one is very small, $1-\exp(-c\ e^{-d}) \ll 1$. For large values of c and d, this effect is only relevant if $c \sim \exp(d)$.

For d-uniform hypergrath with Poisson vertex degree distribution there is a simple relation between the critical mean degree (percolation threshold) and the mean hyperedge cardinality:

$$c^* = \frac{e}{d-1}, \tag{27}$$

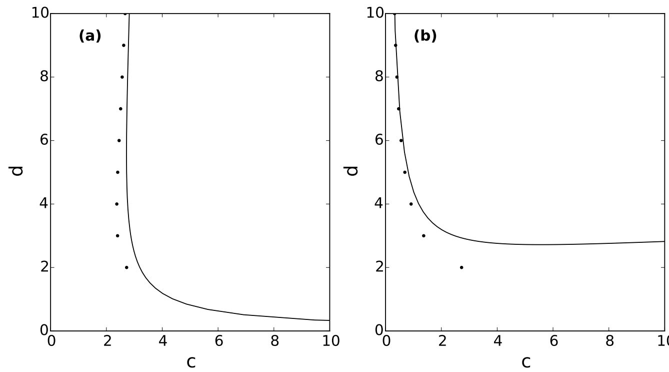

FIG. 6. Phase diagram of the core percolation on hypergraphs with Poisson vertex degree distributions. Black circles and black line represenst the phase boundary of d-uniform hypergraphs and hypergraphs with Poisson hyperedge cardinality distribution, respectivly. (a) core1. (b) core2.

where $e = 2.71828\cdots$ (See Ref. [29] or SI Sec.I for details). The relation between the critical mean degree c^* (percolation threshold) and the hyperedge mean cardinality d is represented in Fig. 6. The phase space of core2 is equal to the phase space of core1 if we interchanged the mean cardinality d with the mean degree c. This is true of course, because as mentioned before, the core2 of a hypergraph is the core1 of the dual hypergraph.

V. CONCLUSION

Hypergraphs are natural generalizations of graphs. The percolation problems on hypergraphs have much richer phenomena than in graphs. For example, we find there are two meaningful hypergraph cores related with two classical combinatorial optimization problems in hypergraphs, i.e., the hyperedge cover and vertex cover problems, respectively. We show that the emergence of the $(2, S)$-core strongly depends on the threshold S, i.e., it can emerge as either a continuous or a hybrid phase transition with different critical exponents. The heterogeneity of vertex degree and the hyperedge cardinality distributions is not fully explored here, which in principle could offer more interesting phenomena.

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Author Information. The authors declare no competing financial interests. Correspondence

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[1] J. Ignacio Alvarez-hamelin, Alain Barrat, and Alessandro Vespignani. Large scale networks fingerprinting and visualization using the k-core decomposition. In Advances in Neural Information Processing Systems 18, pages 41–50. MIT Press, 2006.

[2] Nikhil Bansal and Subhash Khot. Inapproximability of hypergraph vertex cover and applications to scheduling problems. In Proceedings of the 37th International Colloquium Conference on Automata, Languages and Programming, ICALP'10, pages 250–261, Berlin, Heidelberg, 2010. Springer-Verlag.

[3] M. Bauer and O. Golinelli. Exactly solvable model with two conductor-insulator transitions driven by impurities. Phys. Rev. Lett., 86:2621–2624, Mar 2001.

[4] Béla Bollobás and Oliver Riordan. Erdős Centennial, chapter The Phase Transition in the Erdős-Rényi Random Graph Process, pages 59–110. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.

[5] Alain Bretto. Hypergraph Theory: An Introduction. Springer Publishing Company, Incorporated, 2013.

[6] Davide Cellai, Aonghus Lawlor, Kenneth A. Dawson, and James P. Gleeson. Tricritical point in heterogeneous k-core percolation. Phys. Rev. Lett., 107:175703, Oct 2011.

[7] Colin Cooper. The cores of random hypergraphs with a given degree sequence. Random Structures and Algorithms, 25(4):353–375, 2004.

[8] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes. k-core organization of complex networks. Phys. Rev. Lett., 96:040601, Feb 2006.

[9] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes. Critical phenomena in complex networks. Rev. Mod. Phys., 80:1275–1335, Oct 2008.

[10] P. Erdös and A. Rényi. On random graphs, I. Publicationes Mathematicae (Debrecen), 6:290–297, 1959.

[11] M. R. Garey and D. S. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness (Series of Books in the Mathematical Sciences). W. H. Freeman, first edition edition, 1979.

[12] Christina Goldschmidt. Critical random hypergraphs: the emergence of a giant set of identifiable vertices. Ann/ Prob, pages 1573–1600, 2005.

[13] Alexander K Hartmann and Martin Weigt. Statistical mechanics of the vertex-cover problem. Journal of Physics A: Mathematical and General, 36(43):11069, 2003.

[14] Richard M. Karp and M. Sipser. Maximum matching in sparse random graphs. In Foundations of Computer Science, 1981. SFCS '81. 22nd Annual Symposium on, pages 364–375, Oct 1981.

[15] Maksim Kitsak, Lazaros K. Gallos, Shlomo Havlin, Fredrik Liljeros, Lev Muchnik, H. Eugene Stanley, and Hernan A. Makse. Identification of influential spreaders in complex networks. Nat Phys, 6(11):888–893, 11 2010.

[16] Steffen Klamt, Utz-Uwe Haus, and Fabian Theis. Hypergraphs and cellular networks. PLoS computational biology, 5(5):e1000385, 2009.

[17] L. Krumov, C. Fretter, M. Mller-Hannemann, K. Weihe, and M.-T. Houtt. Motifs in co-authorship networks and their relation to the impact of scientific publications. The European Physical Journal B, 84:535–540, 2011.

[18] F. R. Kschischang, B. J. Frey, and H.-A. Loeliger. Factor graphs and the sum-product algorithm. IEEE Trans. Inf. Theory, 47:498–519, 2001.

[19] E. Lawler. Combinatorial Optimization - Networks and Matroids. Holt, Rinehart and Winston, New York, 1976.

[20] Eugene Lawler. Combinatorial Optimization : Networks and Matroids. Dover Publications, March 2001.

[21] Yang-Yu Liu, Endre Csóka, Haijun Zhou, and Márton Pósfai. Core percolation on complex networks. Phys. Rev. Lett., 109:205703, Nov 2012.

[22] Yang-Yu Liu, Jean-Jacques Slotine, and Albert-Laszlo Barabasi. Controllability of complex networks. Nature, 473(7346):167–173, May 2011.

[23] Marc Mezard and Andrea Montanari. Information, Physics, and Computation. Oxford University Press, Inc., New York, NY, USA, 2009.

[24] Michael Molloy and Bruce Reed. A critical point for random graphs with a given degree sequence. Random Structures and Algorithms, 6:161–179, 1995.

[25] M. E. J. Newman, S. H. Strogatz, and D. J. Watts. Random graphs with arbitrary degree distributions and their applications. Phys. Rev. E, 64:026118, Jul 2001.

[26] Robert O'Callahan and Jong-Deok Choi. Hybrid dynamic data race detection. SIGPLAN Not., 38(10):167–178, June 2003.

[27] E. Ramadan, A. Tarafdar, and A. Pothen. A hypergraph model for the yeast protein complex network. In Parallel and Distributed Processing Symposium, 2004. Proceedings. 18th International, pages 189–196, April.

[28] Vladimír Repiský. Covering edges of a hypergraph: complexity and applications. Master's thesis, Comenius University, Faculty of Mathematics, Physics and Informatics, 2007.

[29] Satoshi Takabe and Koji Hukushima. Minimum vertex cover problems on random hypergraphs: Replica symmetric solution and a leaf removal algorithm. Phys. Rev. E, 89:062139, Jun 2014.

[30] Carla Taramasco, Jean-Philippe Cointet, and Camille Roth. Academic team formation as evolving hypergraphs. Scientometrics, 85:721–740, 2010.