Classical Holographic Codes

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ABSTRACT: In this work, we introduce classical holographic codes. These can be understood as concatenated probabilistic codes and can be represented as networks uniformly covering hyperbolic space. In particular, classical holographic codes can be interpreted as maps from bulk degrees of freedom to boundary degrees of freedom. Interestingly, they are shown to exhibit features similar to those expected from the AdS/CFT correspondence. Among these are a version of the Ryu-Takayanagi formula and intriguing properties regarding bulk reconstruction and boundary representations of bulk operations. We discuss the relation of our findings with expectations from AdS/CFT and, in particular, with recent results from quantum error correction.

Contents

1   Introduction

The holographic principle is the statement that a gravitational theory describing a region of space (the bulk) is equivalent to a (non-gravitational) theory confined to the boundary of that region [1, 2]. That is, intrinsically non-geometric features can be equivalently described geometrically. An explicit and very well understood example is the AdS/CFT correspondence [3]. It relates (quantum) gravity on $(d+1)$-dimensional asymptotically Anti-de Sitter (AdS) space to a d-dimensional conformal field theory (CFT) on the boundary. One remarkable aspect of this duality is the interplay of geometry and entanglement that is most evident in the proposal by Ryu and Takayanagi that entanglement entropy in the CFT is equivalently given by the area of a minimal surface in the AdS geometry [4, 5]. This is known as the Ryu-Takayanagi (RT) formula.

Since then, many more connections between geometry and entanglement have been proposed [6–11]. Also, more generally, concepts of quantum information theory were fruitfully applied to gravity and, in particular, to black holes [12–18]. Recently, tensor networks – a tool originally from condensed matter physics to efficiently represent quantum many-body states, especially their entanglement structure [19] – were employed to describe holography [20] and, in particular AdS/CFT [6, 21]. Furthermore, similarities between the properties of bulk operator reconstruction in AdS/CFT and properties of certain quantum error-correcting codes were reported in [22]. There, it is argued that operator reconstruction properties of AdS/CFT are captured by the fact that bulk logical operations can be described by multiple operations on the boundary. Implementing these ideas, an interesting family of toy models for holography was proposed in [23]. There, the authors combine tensor

networks and quantum error-correcting codes. AdS space is tiled with perfect tensors that build up a holographic code and establish an isometric tensor from the bulk to the boundary. These holographic quantum error-correcting codes reproduce some of the key features of the AdS/CFT correspondence, as *e.g.* the RT formula and remarkable bulk reconstruction properties. Later, it was pointed out that a version of the Ryu-Takayanagi formula holds quite generically in quantum error-correcting codes [24]. Furthermore, networks of random tensors were considered [25] and issues like sub-AdS locality [26] and the relation to gauge invariance [27] were addressed. All these constructions are intrinsically quantum and focus on the structure of entanglement.

In this work, we pose the question how far one can get without quantum correlations, like entanglement. Or to put it differently, which features can be reproduced in classical codes that are defined on similar networks? Interestingly, we are able to produce features similar to many of those we mentioned above.

First, motivated by the qutrit example provided in [22], we consider a classical encoding for trits, where one logical (bulk) trit is probabilistically encoded in three physical (boundary) trits. This code has the properties that a "version of the RT formula" for the mutual information holds, the bulk trit can be reconstructed from any two of the boundary trits and logical operations on the bulk trit can be represented by operations on any two of the boundary trits, *i.e.*, there is a notion of subregion duality. Therefore, key features of the AdS/CFT correspondence are captured qualitatively[1] by this example.

Motivated by this example, we then construct a classical code on a network defined by a uniform tiling of hyperbolic space, inspired by the holographic quantum error-correcting codes of [23]. We reproduce many of the features of this quantum code but phrased in a classical language. Although the code is classical, it is not deterministic. We choose probabilistic mappings at one class of vertices of the network and therefore the full mapping from bits in the bulk to bits on the boundary is probabilistic, too. The code produces entropy and classical correlations, where we focus on the latter. For example, we compare the result for the classical mutual information – a measure of correlations – of a finite interval on the boundary with the result for the quantum mutual information. We find that in our classical examples a version of the RT formula holds. That is, the mutual information of an interval on the boundary and its complement is directly proportional to the length of the corresponding minimal surface in the bulk. On the one hand, it might be suspected that the mutual information scales like the area of the minimal surface, since the entanglement entropy measures both, classical and quantum correlations, and it scales with the area of this minimal surface. On the other hand, it is, a priori, not clear, as we do not require any "quantumness" at all to produce the result.[2] This points to the fact that the structure of

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[1]By "qualitatively" we mean that, for example, we cannot represent general quantum operators, as the system is classical. However, we can implement all classical logical operations on the bulk trit by acting on a subset of the boundary trits.

[2]One might argue that the randomness required to generate a probabilistic mapping is reminiscent of quantum superpositions. However, in the case we study, there is no need for any quantum correlations and the randomness in any probability distribution could, in principle, be interpreted as arising from some quantum superposition.

all correlations, classical and quantum, is encoded in the underlying geometric structure.

We also investigate the reconstruction of bulk bits from the knowledge of subsets of the boundary bits and the representation of bulk logical operations on the boundary. Bit flips on a single bulk bit correspond to non-local operations on the boundary. Furthermore, there exists a notion of subregion duality. Therefore, we find a remarkable similarity to the results from quantum codes modeling holography.

The rest of this work is organized as follows. In section 2, we briefly review the properties of the qutrit error-correcting code introduced in [28] and the holographic pentagon code of [23]. Next, in section 3, we introduce *classical holographic codes*. We begin by analyzing a probabilistic trit code that resembles many AdS/CFT-like features, in section 3.1. Subsequently, in section 3.2, we study a network, where each vertex is interpreted as a probabilistic mapping. In particular, we prove a version of the RT formula for the mutual information, the possibility of bulk reconstruction from regions on the boundary, the representation of bulk operations on the boundary, and subregion duality. Finally, in section 4, we give the conclusions of this work.

2   Holographic quantum error-correcting codes

2.1   Qutrit example

In this section, we briefly review a very simple toy model for the AdS/CFT correspondence that is based on quantum error correction. It is formulated as a qutrit[3] code that encodes one logical qutrit into three physical ones such that the logical qutrit can be reconstructed even if one of the physical ones is lost. The key idea is to identify the bulk degrees of freedom with logical qutrits and the boundary degrees of freedom with the physical qutrits [22]. The logical qutrit $|\tilde{\psi}\rangle$ is encoded as

$$
\begin{aligned}
|\tilde{0}\rangle &= \frac{1}{\sqrt{3}} \left(|000\rangle + |111\rangle + |222\rangle \right), \\
|\tilde{1}\rangle &= \frac{1}{\sqrt{3}} \left(|012\rangle + |120\rangle + |201\rangle \right), \\
|\tilde{2}\rangle &= \frac{1}{\sqrt{3}} \left(|021\rangle + |102\rangle + |210\rangle \right),
\end{aligned}
\tag{2.1}
$$

where we indicated the logical qutrit by a tilde to distinguish it from the physical ones [28]. That is, the logical qutrit is encoded in a subspace of the larger Hilbert space of three qutrits, where the code subspace is spanned by the GHZ-type states (2.1),

$$
|\tilde{\psi}\rangle = \sum_{i=0}^{2} c_i |\tilde{i}\rangle.
\tag{2.2}
$$

In consequence, none of the physical qutrits can carry any information about the encoded state, as its reduced density matrix is maximally mixed. However, interestingly, from any

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[3]A qutrit is very similar to a qubit. However, there is one additional base vector spanning its Hilbert space. Therefore, the qutrit state is described by $|\psi\rangle = \sum_{i=0}^{2} c_i |i\rangle$.

two physical qutrits, we denote them by A, B and C, the logical one can be reconstructed. That is due to the existence of operators U_{IJ}, where $I, J = A, B, C$, acting non-trivially only on two of the physical qutrits such that
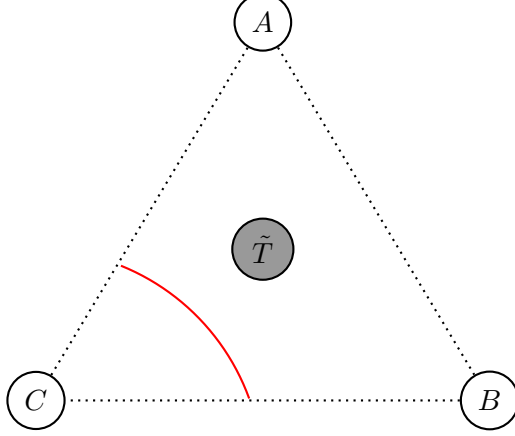

Figure 1. Simplistic toy model for the AdS/CFT correspondence. One logical qutrit $\tilde{T}$ (representing the bulk degrees of freedom) is encoded in three physical qutrits A, B and C (representing the boundary degrees of freedom). The red line sketches the Ryu-Takayanagi surface in the bulk. The logical qutrit can be reconstructed from any two of the bundary qutrits, while only one of these contains no information about it. Furthermore, logical operations on $\tilde{T}$ can also be performed by acting on only two of the physical qutrits. These features are also captured in a classical version of this code we introduce in section 3.1.

$$U_{IJ}|\tilde{i}\rangle = |i\rangle_I \otimes |\chi\rangle_{JK} \,, \qquad |\chi\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |11\rangle + |22\rangle\right) . \qquad (2.3)$$

Therefore it is clear that access to any two qutrits out of the three ($I, J, K \in \{A, B, C\}$) suffices to learn about the logical qutrit. One simply acts on these two physical qutrits with the operator U_{IJ} and obtains qutrit I in the state $|i\rangle$ of the logical qutrit. From this it follows that the action of a logical operator $\tilde{O}$, acting as $\tilde{O}|\tilde{i}\rangle = \sum_j \tilde{O}_{ji}|\tilde{j}\rangle$, can be achieved by the action of a corresponding operator O_{IJ} acting non-trivially on any two physical qutrits. It is of the form

$$O_{IJ} = U_{IJ}^\dagger O_I U_{IJ} \,, \qquad (2.4)$$

where O_I denotes an operator acting solely on qutrit I such that $O_{IJ}|\tilde{i}\rangle = \sum_j \tilde{O}_{ji}|\tilde{j}\rangle$. That is, any logical operation $\tilde{O}$ on the logical qutrit can be performed by acting with the corresponding O_{IJ} on any two physical qutrits. As it was pointed out in [22], this models "subregion duality" in AdS/CFT. Furthermore, this simple toy models obeys a version of the RT formula [24], as we demonstrate next.

As it is clear from above, an arbitrary (mixed) state $\tilde{\rho}$ on the code subspace can be written as

$$\tilde{\rho} = U_{AB}\Big(\rho_A \otimes |\chi\rangle\langle\chi|_{BC}\Big)U_{AB}^\dagger \,. \qquad (2.5)$$

Interpreting the physical qutrits A, B and C as boundary degrees of freedom, we can calculate the entanglement entropy between regions (here: points) in the boundary, see
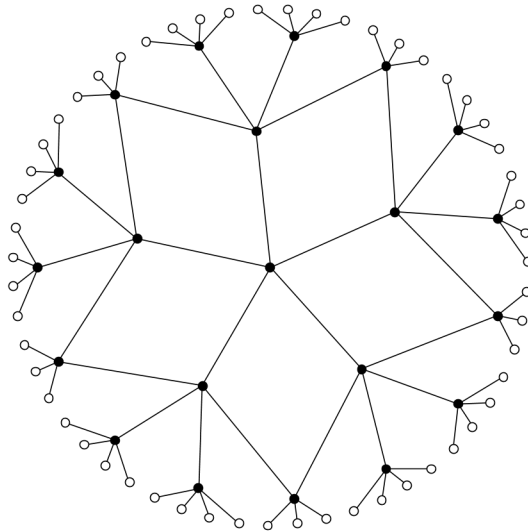
Figure 2. Holographic pentagon code. AdS space is tiled with perfect tensors. Each tensor in the bulk takes one qubit as input (represented as black dots). The boundary contains the outputs of the network (represented by white dots). The network of perfect tensors establishes an isometry from the bulk Hilbert space to the boundary Hilbert space and provides a toy model for the AdS/CFT correspondence.

figure 1. From (2.5), one easily obtains the entanglement entropies

$$
\begin{aligned}
S(\tilde{\rho}_C) &= \log(3)\,, \\
S(\tilde{\rho}_{AB}) &= \log(3) + S(\tilde{\rho})\,,
\end{aligned}
\tag{2.6}
$$

where $\tilde{\rho}_C$ and $\tilde{\rho}_{AB}$ are the reduced density matrices of qutrits C and AB, respectively. That fulfills the RT-formula with area operator $\log(3)$ [24]. Closely related to entanglement entropy is the mutual information that is, in the present case, given by

$$
I_{qu}(C, AB) = S(C) + S(AB) - S(C, AB) = 2\log(3)\,.
\tag{2.7}
$$

The mutual information, however, does not capture contributions from the bulk entropy. Therefore, restricting the states of the boundary qutrits to the class of pure states it is evident that the RT formula can be stated in terms of the mutual information $I_{qu}(A, A^c)$. In this form the RT formula states that the mutual information between a boundary region A and its complement A^c is given by two times the area of the minimal surface in the bulk.

2.2 Holographic pentagon code

The ideas outlined in the previous section led to the investigation of extended networks of concatenated quantum error-correcting codes [23, 25]. Here, we restrict ourselves to the holographic pentagon code, see figure 2, introduced as a toy model for AdS/CFT in [23] and briefly outline some of the ideas behind its construction.

The basic building block of the networks of [23] are perfect tensors. These are defined as tensors $T_{a_1, a_2, \ldots, a_{2n}}$ with the property that they are proportional to isometric tensors

from A to A^c for all subsets A of the tensor indices with $|A| \leq |A^c|$. In particular, perfect tensors are related to quantum states of $2n$ v-dimensional spins as

$$|\psi\rangle = \sum_{a_1,a_2,\ldots,a_{2n}} T_{a_1 a_2 \ldots a_{2n}} |a_1 a_2 \ldots a_{2n}\rangle. \tag{2.8}$$

These states $|\psi\rangle$ have the special property that they are maximally entangled along any possible bi-partition into sets of n spins and therefore show a very particular entanglement structure. Interpreted as a map from one spin to the remaining $2n-1$ spins, a perfect tensor establishes the encoding map of a quantum error-correcting code. It encodes one logical spin into $2n - 1$ spins and allows the recovery of the logical one even if up to $n - 1$ spins are lost. One explicit example for a perfect quantum error-correcting code that gives rise to a state of the kind described in (2.8) is given by the five qubit code in [29]. The qutrit code described in the previous section provides a further example.

For the construction of the holographic pentagon code, the key idea is to tile AdS space with perfect tensors such that a holographic structure emerges, see figure 2. The tensor network describes an isometric tensor from the bulk (the inputs of the tensor network) to the boundary (its output) and can be seen as a quantum error-correcting code that maps the logical qubits (the bulk qubits) to the physical qubits (the qubits of the boundary). Interestingly, in this network the lattice RT formula holds (see (3.12)). Furthermore, the representation of bulk logical operators on different regions of the boundary is analogous to the reconstruction of bulk operators from CFT operators on the boundary. In consequence, this model captures these important features of the AdS/CFT correspondence.

3 Classical holographic codes

In this section, we introduce *classical holographic codes*. These are constructed similarly to the holographic quantum error-correcting codes considered in [23]. Spacetime with non-negative curvature is tiled by a network of maps, where each of the maps is a classical probabilistic one. Furthermore, we impose some constraints on these, as described in section 3.2. We mainly focus on one particular example, however, there are many different constructions possible using, for example, different tilings or trits instead of bits. It is possible to think about the whole network as a classical error-correcting code. However, we do not refer to our construction as an error-correcting code.[4] Besides introducing classical holographic codes, we also discuss their features and find some similarities with expectations from AdS/CFT. In particular, we elaborate on close similarities with quantum error-correcting codes that have recently been considered as toy models for AdS/CFT [22–25].

3.1 Classical trit example

To start our discussion on classical holographic codes, we introduce a classical probabilistic code that resembles key features of the quantum case discussed in 2.1. As in this case, we consider an encoding of a logical trit into three physical ones. Furthermore, we require that

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[4]The reason for this is that probabilistic error-correcting codes are not very widely used in classical coding and we wish to avoid confusions.

the information about the logical trit is zero in each of the individual physical trits, while the knowledge of two of the physical trits provides us with full knowledge about the logical one. One particular code satisfying these constraints is

$$
\begin{aligned}
\tilde{0} &\rightarrow p(000) = p(111) = p(222) = \frac{1}{3}\,, \\
\tilde{1} &\rightarrow p(012) = p(120) = p(201) = \frac{1}{3}\,, \\
\tilde{2} &\rightarrow p(021) = p(102) = p(210) = \frac{1}{3}\,,
\end{aligned}
\tag{3.1}
$$

where $p(X_1 X_2 X_3)$ denotes the probability that the trit string $X_1 X_2 X_3$ ($X_i \in \{0,1,2\}$) appears. In the encoding (3.1), each of the strings has the same probability given by $\frac{1}{3}$. That is, we encode one logical trit in three physical trits in such a way that the logical one is mapped to three different strings of three trits with equal probability. One can convince oneself that the knowledge of one physical trit does not give any information about the logical one, while by knowing any two physical trits we can obtain the logical one with certainty. Labeling the physical trits by A, B and C, as above, that implies that the logical trit can be obtained from either AB, AC or BC, but not from A, B or C alone. That establishes a subregion duality equivalent to the one in the quantum case.

These properties are also reflected in the Shannon entropy S_S. For any of the physical trits I the entropy is given by

$$
S_S(I) = -\sum_i p_i \log(p_i) = \log(3)\,,
\tag{3.2}
$$

where $I = A, B, C$ and the p_i are given by the respective marginal probability distributions. That implies that there is no information about the logical trit in any of the physical ones, as we stated above. Considering any of the sets AB, AC or BC, we find

$$
\begin{aligned}
S_S(IJ) &= -\sum_{ij} p_i \tilde{p}_j \log(p_i \tilde{p}_j) \\
&= \log(3) + S_S(\tilde{I})\,,
\end{aligned}
\tag{3.3}
$$

where $I, J = A, B, C$, the p_i are the probabilities appearing in (3.1) and the $\tilde{p}_j$ give the probabilities for the logical trit $\tilde{I}$ to be $\tilde{X}$ ($\tilde{X} \in \{\tilde{0}, \tilde{1}, \tilde{2}\}$) and we used $\sum_i p_i = \sum_j \tilde{p}_j = 1$. First, we notice that these results are formally the same as in the quantum case discussed in 2.1. That is, a RT formula – at least formally – holds. However, the RT formula is concerned with entanglement entropy, while here we considered the Shannon entropy. To connect both, we move to the mutual information that, for pure states, is equal to two times the entanglement entropy. We find that the mutual information I_{cl} between one physical trit A and the remaining two is given by

$$
I_{cl}(A, BC) = S_S(A) + S_S(BC) - S_S(ABC) = \log(3)\,,
\tag{3.4}
$$

where we used $S_S(BC) = S_S(ABC) = \log(3) + S_S(\tilde{I})$. Due to the symmetry of the encoding the same statement also holds for the other two trits B and C. That is, the classical mutual

information is smaller than the one in the quantum case, (2.7), by a factor of $\frac{1}{2}$. However, it also is proportional to the "area" of the minimal cut.

Let us next investigate whether we can implement logical operations in the bulk (*i.e.*, on the logical trit) by acting on a subset of the boundary degrees of freedom (the physical trits), see figure 1. First, let us implement an operation that implements addition by $\oplus 1$ by solely acting on the physical trits B and C.[5] The operation that succeeds in this task is to apply $\oplus 1$ to B and $\oplus 2$ to C. The same operation can be implemented on A and B by applying $\oplus 1$ to A and $\oplus 2$ to B. Finally, to implement it on A and C, one has to apply $\oplus 2$ to A and $\oplus 1$ to C. To perform the logical operation $\oplus 2$ by acting on two of the physical trits, one has to either act with $\oplus 2$ on B and $\oplus 1$ on C, with $\oplus 2$ on A and $\oplus 1$ on B or with $\oplus 1$ on A and $\oplus 2$ on C. Therefore, operators acting on the logical trit can be reconstructed on either AB, AC or BC, but not on A, B or C alone.

In summary, the classical code we considered shares essential features with the quantum code that we reviewed in section 2.1.

Furthermore, it is interesting to note that the encoding (3.1) can be obtained from (2.1) by imposing complete decoherence.[6] Mapping the classical logical trit given by $\tilde{i}$ ($\tilde{i} \in \{\tilde{0}, \tilde{1}, \tilde{2}\}$) to the logical qutrit state $|\tilde{i}\rangle$ and subsequent encoding according to (2.1), we obtain

$$\rho_{\tilde{i}} = |\tilde{i}\rangle\langle\tilde{i}| = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 & 0_{1\times 6} \\ 1 & 1 & 1 & 0_{1\times 6} \\ 1 & 1 & 1 & 0_{1\times 6} \\ 0_{6\times 1} & 0_{6\times 1} & 0_{6\times 1} & 0_{6\times 6} \end{pmatrix} \tag{3.5}$$

in a basis containing the qutrit states appearing in (2.1), where we denote the basis by $\{|v_j\rangle\}_{j=1,\dots,9}$. Removing the coherences in $\rho_{\tilde{i}}$, for example, by a randomly selected projective measurement with projectors $P_j = |v_j\rangle\langle v_j|$, we arrive at a mixed state $\rho_{\tilde{i}}^{(dec)} = \frac{1}{3}\sum_{j=1}^{3}|v_j\rangle\langle v_j|$. This is a statistical mixture of pure states $|v_j\rangle\langle v_j|$ that appear with probability $p(v_j) = \frac{1}{3}$. Therefore, by reinterpreting the qutrits as classical trits, we obtain the encoding (3.1).

At this point, we would like to insert another brief comment. There is the question how the randomness in the description of the system can be justified physically. In our opinion, there are (at least) three possible ways. One is that there is a lack of knowledge about the details of the system that forces a probabilistic description, like in thermodynamics. Another way to justify the randomness in the code is to imagine an agent at each vertex that generates the randomness that is necessary for the functioning of the code, for example, by sending individual photons to a beam splitter and subsequently collapsing the quantum superposition of the photons. In this way the agent can create the required random numbers. Similarly, one could think of strong local decoherence at each of the vertices that kills the

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[5]Here and in the remainder of this section, $\oplus n$ for some integer n denotes the addition by n mod3.

[6]Note that the classical encoding (3.1) neither does have to be obtained in this way nor does it have to be interpreted in this way. Also, already at this point, we want to mention that the classical codes on extended networks, we introduce in the next section, cannot be obtained by decoherence of the boundary state of *e.g.* the holographic pentagon code.
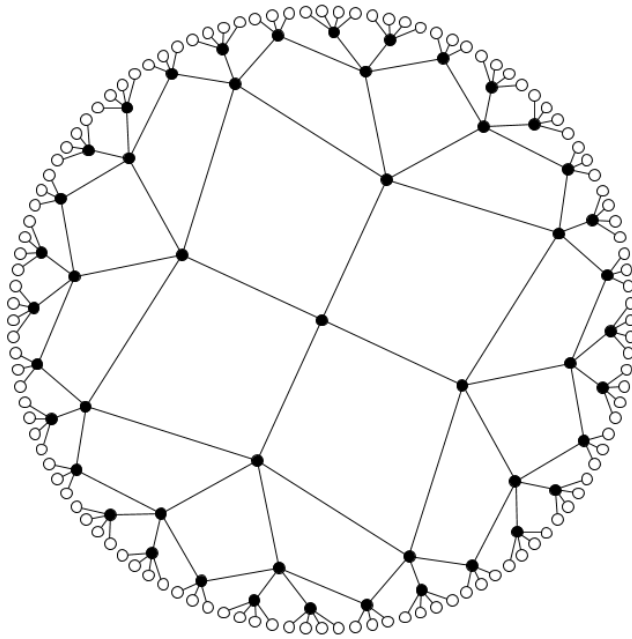
Figure 3. Network to realize a classical holographic code. Each vertex in the interior of the graph represents a tile with a specific fixed volume in AdS space. Furthermore, each of these vertices takes one bit as input (the input bits are then interpreted as bulk degrees of freedom) and (probabilistically) maps the input together with the input from the in-going edges to the out-going edges. The final output of the code is then given by the bits sitting at the boundary of the network. These are interpreted as boundary degrees of freedom. In this way a map from bulk degrees of freedom to boundary degrees of freedom is established that gives rise to a duality between bulk and boundary.

coherences and leaves us with a probabilistic mixture, as described above. However, in our opinion, it also is enough to just state that the codes we consider are intrinsically random.

3.2 Classical codes on hyperbolic space

We study a classical probabilistic code on a tiling of AdS space that features some key properties of the quantum codes [22–25] under which there are the Ryu-Takayanagi formula and important bulk reconstruction properties. The tiling gives rise to a network as *e.g.* visible in figures 2 and 3. Via the network we define a (probabilistic) mapping from the bits sitting on the vertices in the interior to those on the open edges at the boundary. The mapping is defined as follows: We order the network into layers of vertices defined by the graph distance from the center. From the negative curvature of the graph it follows that each vertex shares at most two edges with vertices of the previous layer. We now declare each node to a map $n \to m$, where n is the number of inputs given by the bit at the vertex and edges from the previous layer, and m is the number of output bits. The three possible mappings appearing in this tiling are $3 \to 3$, $2 \to 4$, and the one in the center. In particular, we claim the following feature for all the mappings

(I) the knowledge of three output bits gives complete information about the other bits.

Due to this property the $3 \rightarrow 3$ map must be well-defined, *i.e.*, non-probabilistic, and, hence, it is bijective. An exemplary map is given by

$$
\begin{aligned}
0_{e_1}\tilde{0}0_{e_2} &\rightarrow 101\,, \quad 0_{e_1}\tilde{0}1_{e_2} \rightarrow 011\,, \quad 0_{e_1}\tilde{1}0_{e_2} \rightarrow 111\,, \quad 0_{e_1}\tilde{1}1_{e_2} \rightarrow 001\,, \\
1_{e_1}\tilde{0}0_{e_2} &\rightarrow 110\,, \quad 1_{e_1}\tilde{0}1_{e_2} \rightarrow 000\,, \quad 1_{e_1}\tilde{1}0_{e_2} \rightarrow 100\,, \quad 1_{e_1}\tilde{1}1_{e_2} \rightarrow 010\,,
\end{aligned}
\tag{3.6}
$$

where the tilde indicates the bulk input bit and e_1, e_2 denote the bits from the incoming edges. The above map produces no entropy but classical correlations. It has the additional important property, that the knowledge of only one bit gives no information about the bulk input. It is also important that (IIa) the knowledge of any single edge bit gives no information about neighboring edge bits.[7]

For the two other possible maps we demand the even stronger property that

(IIb) The knowledge of a single edge bit gives now information about any other bit.

The $2 \rightarrow 4$ mapping cannot be well-defined if it should fulfill the above properties, *i.e.*, it has to be a probabilistic map: the output is given by some classical probability distribution. The explicit example we choose is

$$
\begin{aligned}
0_e\tilde{0} &\rightarrow p(0000) = p(1111) = \frac{1}{2}\,, \quad 0_e\tilde{1} \rightarrow p(0110) = p(1001) = \frac{1}{2}\,, \\
1_e\tilde{0} &\rightarrow p(0101) = p(1010) = \frac{1}{2}\,, \quad 1_e\tilde{1} \rightarrow p(1100) = p(0011) = \frac{1}{2}\,,
\end{aligned}
\tag{3.7}
$$

where the second input bit is the bulk input, e denotes the input from the incoming edge, and $p(X_1 X_2 X_3 X_4)$ denotes the probability of the output $X_1 X_2 X_3 X_4$ ($X_i \in \{0,1\}$). The map (3.7), additionally, fulfills properties that we need to proof the desired features of classical holographic codes. These are that (i) the knowledge of two edge bits gives at most one other bit with certainty, and (ii) the knowledge of both inputs and one output gives all the other bits. In the example of a $2 \rightarrow 4$ mapping (3.7), there are two possible outputs occurring with the same probability for each input. The amount of entropy it produces is therefore given by $\log(2)$. In consequence, the (Shannon) entropy $S_S^{(out)}$ of the output given a probabilistic input $I_l\tilde{J}$ with entropy $S_S^{(in)} = S_S(I_l) + S_S(\tilde{J})$ is given by

$$
S_S^{(out)} = \log(2) + S_S^{(in)} = \log(2) + S_S(I_l) + S_S(\tilde{J})\,.
\tag{3.8}
$$

Finally, in the interior of the bulk, there is only one more mapping, namely the one in the center of AdS space. So far, we restricted ourselves to a pentagon tiling which implies that every vertex has five neighbors and therefore there are five edges connected to it; see figure 2. That implies for the vertex in the center that it should be a $1 \rightarrow 5$ map, as there is no deeper layer. A possible probabilistic mapping is given by

$$
\begin{aligned}
\tilde{0} &\rightarrow p(00000) = p(01111) = p(10101) = p(11010) = \frac{1}{4}\,, \\
\tilde{1} &\rightarrow p(11100) = p(10011) = p(00110) = p(01001) = \frac{1}{4}\,.
\end{aligned}
\tag{3.9}
$$

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[7]This is important in the proof of the RT formula.

This mapping is sufficient to obtain most of the properties of the network we want to show. However, it breaks the global pentagon symmetry of the configuration. This is because the central bulk bit can be reconstructed with the knowledge of the second and forth output bit but not with the knowledge of any other two bits. Therefore these two bits are distinguished. All $1 \to 5$ mappings break the symmetry in a similar way.

A second possibility is to introduce a single square, instead of a pentagon, in the center. This breaks the pentagon symmetry down to a global square symmetry; see figure 3. However, it is now possible to define a mapping that fulfills all the above properties without further breaking symmetry. An example is given by

$$
\begin{aligned}
\tilde{0} &\to p(0000) = p(1111) = p(1010) = p(0101) = \frac{1}{4}\,, \\
\tilde{1} &\to p(1100) = p(0011) = p(1001) = p(0110) = \frac{1}{4}\,.
\end{aligned}
\tag{3.10}
$$

Both of the above exemplary mappings for the center have the additional property that any two neighboring output bits give no information about any other bit, which again is needed for the properties we want to show for the network. In what follows and if necessary we focus on the second case with the square in the center.

We show that mappings with the outlined properties – and in particular our specific examples – together with the geometric structure of the network are sufficient to produce the above mentioned features. For that reason, we call them *classical holographic codes*. After having introduced classical holographic codes as a set of (probabilistic) mappings residing on a graph on AdS, we next investigate their properties. Therefore, we focus on the tiling shown in figure 3 and consider a particular choice of mappings given by (3.6), (3.7) and, for the center, (3.10). However, the pentagon tiling, figure 2, with the center vertex chosen to be (3.9), admits almost exactly the same features.

3.3 Features of classical holographic codes

In this section, we investigate to what extend our example of a classical probabilistic code defined by a network on AdS produces some properties similar to those of quantum error-correcting codes. As we find, there is a close similarity between the properties of these codes and AdS/CFT. Classical holographic codes model a classical correspondence between bulk and boundary.

3.3.1 Ryu-Takayanagi formula

Consider a CFT with a gravitational dual, where at least for every static state at low energies there exists a geometric bulk description. In these states, the Ryu-Takayanagi (RT) formula[8] relates the entanglement entropy S_A of a boundary region A at fixed time to the area of the minimal surface γ_A in the bulk, whose boundary coincides with the boundary of A

$$
S_A = \frac{\text{Area}(\gamma_A)}{4G}\,,
\tag{3.11}
$$

---

[8]Here and in the following, we do not consider contributions from bulk entropy.

where G is Newton's constant [4, 5].

An analogous relation holds for the quantum error-correcting code considered in [23]. Considering a so-called holographic state – that is a boundary state of a tensor network of perfect tensors with a graph of non-positive curvature – then measured in units of $\log(2)$ the entanglement entropy of any connected region A on the boundary equals the length of the shortest cut[9] γ_A through the network whose boundary matches that of A

$$S_A = |\gamma_A| \,. \tag{3.12}$$

The statement simply is, that for these tensor networks the lattice RT formula holds.

Interestingly, in the case of a classical holographic code a very similar statement is true. Of course, the concept of entanglement entropy does not exist in these classical systems. In particular, there is no quantum entanglement. However, if we interpret this quantity not only as a measure of quantum entanglement but of correlations, or even more abstract as a measure of joint information between two subsystems, then there is a classical analogue namely the mutual information I_{cl}. It can formally be defined in the same way for both classical and quantum theories

$$I_{qu/cl}(A, B) = S(A) + S(B) - S(A, B) \,, \tag{3.13}$$

where A and B denote two subsystems and the subscripts qu and cl specify the quantum mutual information I_{qu}, defined in terms of von Neumann entropies, and the classical mutual information I_{cl}, defined in terms of Shannon entropies. In a quantum theory $S(A)$ and $S(B)$ are the von Neumann entropies of the respective reduced density matrices of subsystems A and B. $S(A, B)$ denotes in this case the von Neumann entropy of the union of A and B. For a bipartition of a system in a pure state into two subsystems A and $B = A^c$, the total entanglement entropy vanishes, $i.e.$, $S(A, B) = 0$, and the two partitions show equal entropy, $S(A) = S(B) \equiv S_A$, such that

$$I_{qu}(A, A^c) = 2S_A \,. \tag{3.14}$$

In a classical system $S(\cdot) \equiv S_S(\cdot)$ denotes the Shannon (or marginal) entropy of the system inside the bracket. As in the quantum case, the mutual information measures the joint information of the two subsystems A and B. However, for classical systems, the mutual information is solely due to classical correlations between subsystems.

The version of the RT formula for the mutual information, in case of classical holographic codes is the following. For an arbitrary but fixed bulk input, the classical mutual information between a (connected) subregion A on the boundary and its complement A^c is given by the length of the minimal cut γ_A through the network, whose boundary matches that of A,

$$I_{cl}(A, A^c) = |\gamma_A| \,. \tag{3.15}$$

---

[9]A cut is nothing but a path through the network that separates it into two disjoint sets of vertices and the length of the cut is given by the number of edges it crosses.

Therefore, one can say, a version of the RT formula holds for these classical systems and measures classical correlations.[10]

Proof of the RT formula (3.15) for classical holographic codes   We first argue that the mutual information of a connected region A and its complement is bounded from above by the length of the minimal cut γ_A, *i.e.*,

$$I_{cl}(A, A^c) \leq |\gamma_A| \,. \tag{3.16}$$

It is evident that all correlations in the system must be generated in the interior of the bulk and are transported by the network to the boundary. If we consider an arbitrary cut through the network whose ends coincide with the boundary of the interval, then all correlations between regions A and A^c are transmitted in the edges that are crossed by the cut. Of course, that is also true for the minimal cut γ_A and, since every edge can at most transfer one bit of information, the amount of correlation (or shared information) is bounded from above by the length of this cut. Therefore, bound (3.16) holds.

In the case of classical holographic codes, the upper bound (3.16) for the mutual information is saturated, as we show next. The general idea of the proof is that any of the bits that are transferred through an edge crossed by the minimal cut γ_A can be reconstructed with certainty from either side. Furthermore, there is no correlation between the edge bits crossed by γ_A. Then each of the bits has to carry on bit of shared information and hence contribute to the mutual information by one. In consequence, the mutual information is given by the length of the minimal cut γ_A and a version of the RT formula (3.15) holds.

One can convince oneself that this statement is true by considering an algorithm for constructing the minimal cut that was also presented in [23]. Given some connected region of the boundary, the algorithm starts with the cut that crosses all the open edges at the boundary. The algorithm then proceeds in the following way: It lets the cut jump over a vertex if at least three edges of one vertex are crossed by the cut. After the jump it crosses all the edges of the vertex that were not crossed before. Then, given the new cut, it starts again. This algorithm stops when the cut is minimal, cf. figure 4. From that it is clear that each bit flowing through any edge crossed by a cut constructed in this way can be reconstructed from the bits of the boundary region it starts from. This directly follows by applying property (I) or (ii) in every step of the algorithm. In most cases the minimal surface constructed from a connected region A on the boundary and the one from its complement coincide. However, as also mentioned in [23] there is the possibility that these do not coincide. In the first case, we certainly can construct the edge bits of the unique minimal surface from both sides.

Next, we argue that the edge bits that are crossed by a unique minimal cut are not correlated. Therefore, we show that no information about an edge bit can be gained by the knowledge of any subset of the remaining ones. First, we assume the opposite, *i.e.*, one can obtain information about a crossed edge bit E from the knowledge of other crossed

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[10]Note that the lattice RT formula (3.12), that was proven for holographic quantum error-correcting codes, can, for pure boundary states, be written in terms of the mutual information, as $I_{qu}(A, A^c) = 2|\gamma_A|$. Thus, it is evident that for quantum codes the mutual information is twice the classical one. That is due to the fact that the quantum mutual information measures both "quantum" and "classical" correlations.
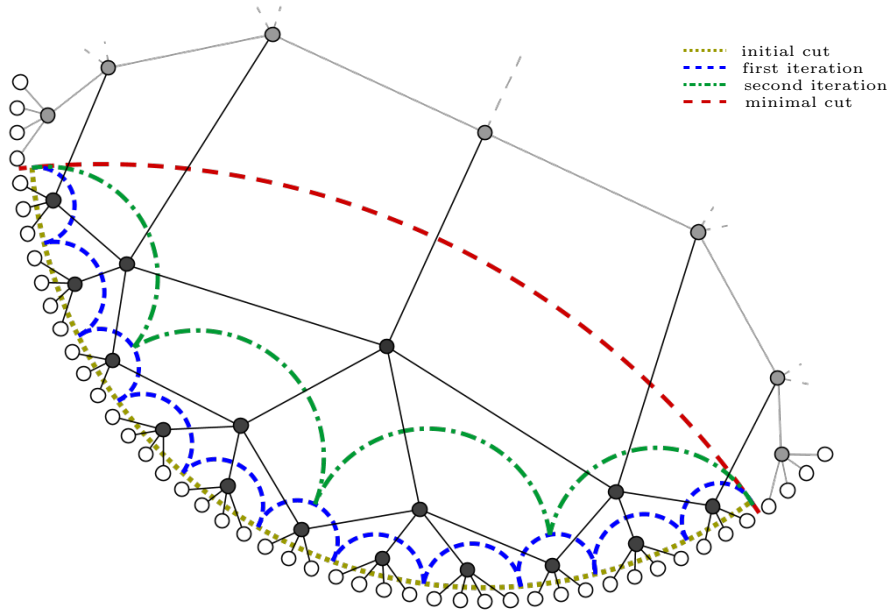
Figure 4. Visualization of the algorithm the constructs the minimal cut (red) for a boundary region. The algorithm starts from a cut that divides the bits in that boundary region form the remaining system (initial cut). Then, for each vertex, it evaluates how many edges belonging to the vertex are crossed by this cut. If this number is larger or equal to three, the cut is moved across the vertex such that it cuts all edges of this vertex that have previously not been crossed (in the first iteration that results in the blue cut). Subsequently, it takes the new cut as starting point. The algorithm terminates, when the cut is minimal (red cut).

edge bits. First, fix the vertex via which one assumes to get information about E. One can convince oneself that, from other edges crossed by the minimal cut, it is not possible to gain information about edges that are connected to this vertex and point "deeper into the bulk", *i.e.*, that are not next to E. This is because the graph distance to those edges is too large. Therefore, due to the geometric structure, no information about E can be obtained from these edges. So we need to get the information via the two neighboring edge bits. We need both because of property (IIa) or (IIb). We can only get information about those if they are crossed known edge bits themselves or if they go "parallelly" to the minimal cut, by which we mean that they again are connected to a vertex connected to a crossed (but known) edge next to E. An analogue reasoning now tells us that again we need the knowledge of a parallel edge bit if not a second known crossed edge bit ends at the same vertex to get information about the parallel edge. This reasoning goes on and has to terminate before the boundary which means there has to be a vertex before the boundary to which two known crossed edge bits are connected. In this scenario each of the vertices connected to parallel edges can have at most two more edges pointing "deeper in the bulk". It directly follows that the minimal cut that is constructed starting from the boundary in the direction of the parallel edges does not jump over these vertices, because it crosses only two of their

edges, *i.e.*, it cannot coincide with the minimal cut that we started with. Hence, we have no unique minimal cut and therefore a contradiction. This reasoning is also illustrated in figure 5. This proves that the edges crossing the minimal cut cannot be correlated. Together with the fact that each edge crossed by the minimal cut can be reconstructed from either side, this finishes the proof of the RT formula (3.15).
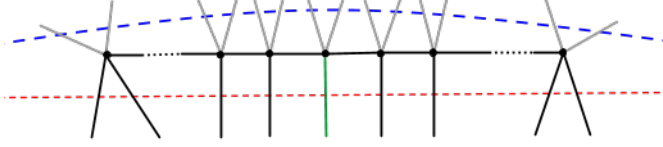
$\square$

Figure 5. Illustration of the reasoning about the correlation of crossed edge bits. We assume the green edge bit through the red minimal cut can be constructed. The black edges are needed to gather information about the green one. The grey edges are pointing away from the minimal cut and can give no information. The blue cut is the minimal cut constructed – using algorithm illustrated in figure 4 – from the complementary boundary. The two minimal cuts do not coincide. Hence, if we assume some bits are correlated the minimal cut cannot be unique.

Note that there are still the cases left, where the minimal surface is not unique. From the argumentation above it becomes clear that for those the mutual information is smaller than the length of the minimal cut.

All these results are supported by numerical checks up to the fourth layer of the network.[11]

3.3.2 Bulk and operator reconstruction

In AdS/CFT, a gravitational theory on $(d+1)$-dimensional asymptotically-AdS space (bulk) is related to a d-dimensional conformal field theory on the boundary. That immediately raises the question how, given some configuration of the boundary, the bulk can be reconstructed. This is, in particular, complicated by the emergent spatial dimension. The information required to reconstruct some region of the bulk is contained in a boundary region if its entanglement wedge contains this region of the bulk [30], see figure 6. Here, in the classical case, we argue that the relevant wedge is the *correlation wedge* $C(A)$ that is defined as the region bounded by the minimal cut. It is therefore very similar to entanglement wedge reconstruction. In the following, we demonstrate the possibility of bulk reconstruction in the *correlation wedge* of a region of the boundary. Furthermore, we address the issue of operator reconstruction and show that – in our case – classical operations, like bit flips on bulk degrees of freedom contained in the correlation wedge of some boundary region A can be performed by acting (non-locally) on the boundary degrees of freedom in A.

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[11]The numerical results also suggest that in the case of two different minimal cuts coming from A and A^c, the mutual information is given by the length of the smaller cut minus the number of connected regions between the two cuts.
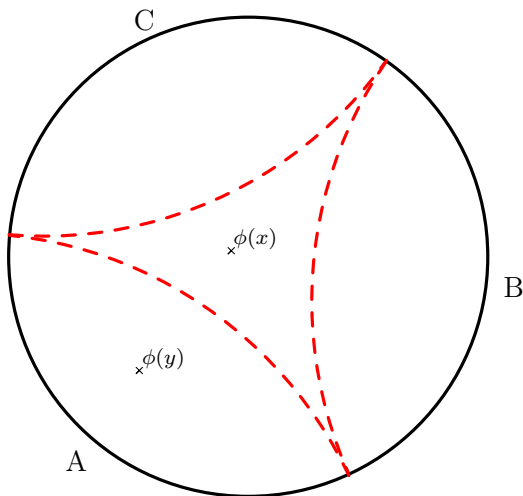
Figure 6. Subregion duality. The operator $\phi(y)$ can be represented on the boundary region A, but not on the complementary region $A^c = BC$. The operator $\phi(x)$, however, cannot be represented on any region A, B or C. But, still, it can be represented on the union of any two of these regions, *i.e.*, on AB, AC and BC. That is referred to as subregion duality.

Let us assume A is connected and the minimal cut γ_A is unique. If we further assume that A is smaller than half of the boundary, then we can reconstruct every bulk input bit in $C(A)$. This is evident by considering the algorithm for constructing the minimal cut. In every step it crosses three outgoing edges that allow to reconstruct all the other bits, including the bulk input of the vertex it jumps over – due to property (I).

If we consider a region A that is larger than half of the boundary then there exists the possibility that two outgoing edges of one vertex in $C(A)$ are crossed by the minimal cut. Then, for a $2 \to 4$ mapping it may be that one cannot reconstruct this particular bulk input. However, all the other inputs can be reconstructed. We do not consider this as a crucial problem, as in the limit of large networks, *i.e*, where the number of bulk inputs goes to infinity, the number of inputs that cannot be reconstructed is negligible. Therefore, the bulk contained in the correlation wedge $C(A)$ can be reconstructed from the respective boundary region A.

Next, we consider the reconstruction of bulk operations.[12] Assuming a connected boundary region A, all bit flip operations O on vertices in the bulk region $C(A)$ can be represented as multiple bit flips in A. The reason for this is the following. Any vertex in region $C(A)$ has by construction at least three neighboring edges that are contained in $C(A)$ and go in the direction of A; see figure 7. Solely flipping some of the these bits cannot affect bits in the complement of A. Therefore, degrees of freedom in A are sufficient to reconstruct operations in $C(A)$. Consider now the action of an operation O on a vertex in $C(A)$. Then it is possible to successively modify the edge bits in $C(A)$ until we reach the boundary

---

[12]Note that in our classical code the "bit flit operator" O is the only non-trivial operation.

region A. Obviously, no edge bit leaving $C(A)$ is touched by this procedure. Therefore the operation O on any bulk bit in $C(A)$ can be reconstructed by flipping the respective subset in A that was flipped by the above procedure. This is, in general, not possible for bit flips on vertices not contained in $C(A)$ but with some exceptions directly behind the minimal cut. The exceptions are most evident when we look at our specific examples. Consider *e.g.* the $3 \to 3$ mapping in (3.6) and a minimal cut that only crosses the outgoing edge in the middle, then it is possible to realize a bulk bit flip from either boundary region. For the $2 \to 4$ mapping as in (3.7) the same is true for the two middle output bits. Again we do not regard this as a crucial problem, for the same reason as above.
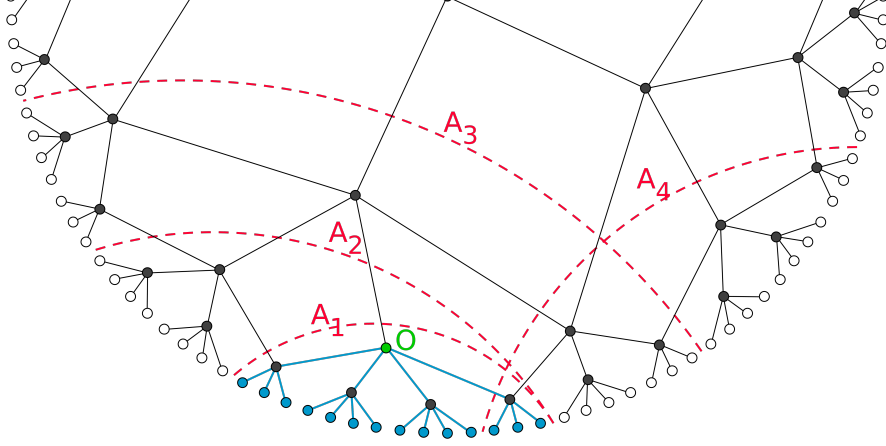

Figure 7. Representing bulk operations. The action of an operation O on one of the bulk bits, bit I, can be definitely represented on a boundary region A_i if I is contained in $C(A)$ (and in some cases if it is directly behind the minimal cut). Here, we show one particular example and marked the edges and vertices blue that can be affected by the operation on the green vertex. Here, $C(A_{1,2,3})$ contain the bit I and hence the bit flip O can be realized on these boundary regions. A_4 is an example that does not allow to reconstruct O.

Another question arising here is, whether the possible operations on the boundary region to realize a specific bit flip in the bulk depend on the configuration of the boundary bits. For the examples given in (3.6) and (3.7) this is not the case. This becomes evident if we look at the single mappings. Flipping some inputs in a specific way always leads to the same possible flips in the output independent of the actual values of the bits. For example, flipping the bulk input in the $3 \to 3$ map always flips the middle output bit, or solely changing the edge input in the $2 \to 4$ mapping can always be realized by changing the first and third output. It never depends on the actual value of the bits. This holds for any mapping in the network, so in total it holds for the whole network.

Subregion duality The so-called subregion duality in AdS/CFT states that operators in the bulk can, in general, be represented on different subregions of the boundary, see figure 6. In [22], the toy model we reviewed in section 2.1 was suggested to capture essential features of this duality. Also in more elaborate tensor network models based on quantum error-correcting codes, it was shown to hold [23]. Here, we show that also in the classical network, we introduced, there is a notion of subregion duality. Indeed, it immedi-

ately follows from the fact that an operation O on any bulk input I can be represented on a boundary region A_i if $I \in C(A_i)$, as we have shown above; also see figure 7. Therefore, all representations of O on each of the A_i's are dual to each other. This establishes a notion of subregion duality for classical holographic codes.

Black holes A naive picture of asymptotically AdS spacetimes containing black holes is to describe these configurations by "cutting out" some region of the network [23]. The microstates of black holes are then described by the edge bits crossed by the horizon that function as inputs for the remaining network. In consequence, the black hole has a non-vanishing entropy that scales like the number of edges crossed by the horizon, *i.e.*, it scales like the area of the black hole. Interestingly, this behavior is only expected in the semi-classical approach [31, 32] and should not appear at the classical level. However, we emphasize that this picture of black holes is very naive.

4 Conclusions

In this work, we introduced *classical holographic codes* and analyzed their properties. Interpreting the input of the codes as the bulk degrees of freedom and its output as the boundary degrees of freedom, a classical holographic code establishes a map between these. One of the main features of the codes is that a version of the Ryu-Takayanagi formula holds; the mutual information between a connected region A on the boundary and its complement A^c is given by the length of the minimal cut γ_A that ends on the boundary of A. We defined the bulk region that is enclosed between γ_A and the boundary region A as the correlation wedge $C(A)$ of A. We have shown that the bulk inputs contained in $C(A)$ can be reconstructed from the data in A. Furthermore, we have shown that a (bit flip) operation O, acting on any bulk input contained in $C(A)$, can be represented by multiple bit flips in the boundary region. Finally, we established a notion of subregion duality. That is, we have shown that any operation O acting on some input in the bulk can be represented in any boundary region A that possesses a respective correlation wedge $C(A)$ such that the bulk input is contained in it.

We did not intend to construct a purely classical toy model for the AdS/CFT correspondence. However, interestingly, all the features we described above are to be expected from AdS/CFT. Furthermore, these are the features that are modeled by quantum error-correcting codes, such as the ones in [22, 23]. Of course, there is the obvious caveat that the boundary theory is purely classical and by no means can approximate a quantum CFT. In particular, the entanglement structure of a quantum CFT is completely absent. Another shortcoming of the classical code is that bulk and boundary operations (bit flips) are rather simple compared to general operators appearing in a CFT. Finally, in our particular example, the center vertex has some shortcomings, as we described. However, especially in the limit of large networks, the center vertex should not cause serious problems.

Even so there are these shortcomings in the construction, it is interesting to note that, by starting from a purely classical code, one can obtain all the AdS/CFT-like features, we outlined above. This shows that, given the geometric structure of the network, the scaling of the mutual information, *i.e.*, a version of the RT formula, and important bulk and operator

reconstruction properties are due to the "correlation structure" and can exist even classically in the absence of quantum correlations, like entanglement.

In summary, the correlations on the boundary seem to be (partially) encoded in the geometric structure of the network in the bulk.

Acknowledgments

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