Environment-assisted analog quantum search

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The coupling of a quantum system to an external environment has been known to improve energy transport efficiency to a reaction center in certain light harvesting systems. The corresponding dynamics is often modeled by a continuous time quantum walk on a disordered structure. Thus it is natural to ask whether such environment-assisted effects can be observed for a quantum algorithm based on quantum walks. In this work, we consider the problem of quantum search of a marked node on a complete graph of n nodes in the presence of static disorder, and show that coupling this system to an external environment can significantly improve the success probability of the algorithm. In particular, we demonstrate that for strong disorder, characterized by a standard deviation σ for each site energy, the presence of a thermal bath increases the success probability from $1/(n\sigma^2)$ to at least $1/2$ as the system relaxes to a steady, thermal state. Remarkably for a fixed σ, the system relaxation time decreases for higher temperatures, within a large temperature regime, but is lower bounded by the corresponding running time of quantum search in the absence of any bath. Furthermore, we discuss for what regimes of disorder and bath parameters quantum speedup is possible. This work suggests that naturally occurring open quantum system dynamics can be advantageous for analog algorithms affected by static errors.

A major obstacle to the development of a scalable quantum computer is its interaction with an environment, resulting in decoherence and loss of quantum advantage [1, 2]. Even if a quantum system is well isolated from the environment, there are always experimental imperfections in the setting of the system's parameters which can lead to a unitary dynamics different from the desired one and thus to errors in the quantum computation. In the circuit model, these sources of error can be countered using various error correction techniques [3]. However, these have proven to be rather expensive as they require a huge overhead in terms of the number of qubits [4]. Furthermore, for alternative models of quantum computation such as adiabatic [5, 6] or quantum walks [7, 8], the theory of error correction is much less developed or nonexistent [9–11].

On the other hand, there are quantum processes that are enhanced by the interaction with an external environment. It has been shown that quantum transport in certain disordered structures like protein complexes in biological systems [12–15] and others [16–18] can be enhanced for certain ranges of environment parameters. A simplified explanation for this mechanism is that in a disordered quantum system there are destructive interferences suppressing quantum transport [19] and since decoherence processes suppresses these destructive interferences, transport efficiency is enhanced. Also, relaxation dynamics coming from the interaction with a thermal bath can significantly improve quantum transport provided that the bath spectral density is in a regime which enhances certain desired transitions [20].

Since quantum transport is described as a continuous-time quantum walk (CTQW) [12] it is natural to ask whether an analogous environment-assisted phenomenon could happen in a quantum walk based algorithm subjected to static errors. In this work, we address this question by considering the analog version of Grover's algorithm [21] which can be seen as an instance of search by CTQW on the complete graph of n nodes [22]. This algorithm finds a node in the graph, which is marked by an oracular Hamiltonian, starting from an equal superposition of all the nodes of the graph, in $\mathcal{O}(\sqrt{n})$ time. This running time is quadratically faster than the best known classical algorithm, and is optimal [21].

We consider the effect of a static diagonal disorder term of strength at most σ in the search Hamiltonian which can be interpreted as a faulty oracle. We show that for $\sigma > \mathcal{O}(1/\sqrt{n})$ the algorithm loses its optimality. Above this threshold, we find that the maximum probability of success decreases with the size of the system and several repetitions are needed to find the marked node.

By coupling the system to a thermal environment [23–25], the transition from the initial state to the marked node, which was suppressed in the unitary case due to disorder, is now enhanced because of thermal relaxation. This is because the dynamics occurs mostly in a two-dimensional subspace spanned by the ground and first excited states of the system, where the ground state has a large overlap with the marked node. So, the system relaxes to a thermal state which has a constant overlap with the solution and hence the algorithm exhibits a fixed-point property. Thus only a constant number of repetitions are needed to find the marked node and a measurement can be made at any time after the system relaxes. Interestingly, the relaxation time and thus the algorithmic running time improves with temperature as long as the

two-level approximation is valid. For the maximum allowed temperature and for a fixed disorder strength σ, the scaling of this relaxation time matches the running time of the corresponding closed system with the same disorder strength, up to logarithmic factors.

Our work contrasts with the idea of engineering the dissipation of a quantum system in order to drive a quantum computation [26]. Instead, we study how a naturally occuring coupling to a thermal bath can help when static errors are present in the system Hamiltonian. This way, our results also differ from those concerning thermal effects in adiabatic quantum computation [27–32].

Before proceeding with a careful analysis of the scaling of the running time with the different bath parameters, let us look at the closed system behavior in the presence of static disorder and analyze the algorithm in that scenario.

Analog quantum search with diagonal disorder.— Let G be a graph with n vertices $V = \{1, 2, .., n\}$. We consider the Hilbert space spanned by the localized quantum states at the vertices of the graph $\mathcal{H} = \text{span}\{|1\rangle, ... |n\rangle\}$ and the search Hamiltonian given by

$$H_{search} = -|w\rangle\langle w| - \gamma A_G, \tag{1}$$

where $|w\rangle$ corresponds to the solution of the search problem, γ is a real number and A_G is the adjacency matrix of graph G [22]. The algorithm is said to be optimal on graph G if starting from the equal superposition of all states, i.e. $|s\rangle = \sum_{i=1}^{n} |i\rangle/\sqrt{n}$, there is a value of γ such that the probability of finding the solution node $|w\rangle$ upon a measurement in the vertex basis after a time $T = \mathcal{O}(\sqrt{n})$ is constant, irrespective of w. Here we consider quantum walk on a complete graph which is equivalent to the analog quantum search algorithm introduced in [21]. The search Hamiltonian in that case is given by

$$H_{search} = -|w\rangle\langle w| - |s\rangle\langle s|, \tag{2}$$

where we have chosen $\gamma = 1/n$. The gap between the ground state and the first excited state, up to an error of $\mathcal{O}(1/n)$ is $\Delta = 2/\sqrt{n}$. The dynamics of the algorithm is a rotation in a two-dimensional subspace containing the initial state $|s\rangle$ and $|w\rangle$. The success probability $P_w(t) = \sin^2(t/\sqrt{n})$ is close to one after a time $T = \pi\sqrt{n}/2$.

The analog search algorithm requires an oracle that marks the solution node to an energy that is different from the rest of the nodes. In order for the problem to have a fair comparison to the standard Grover's algorithm in the circuit model, the energy at the marked node is chosen to be -1 [33]. However, an imperfect implementation of the oracle might severely affect the algorithmic performance. We define an imperfect oracle as one which "marks" each node of the graph erroneously: each non-solution node j is marked with an energy ϵ_j, while the solution node, w is marked with an energy $-1+\epsilon_w$ (where each ϵ_w is a random variable). The resultant effect can be perceived as static disorder on the nodes of the complete graph. Furthermore we assume that these errors occurring due to imperfect implementations are fixed in nature, i.e. each ϵ_i remains fixed across multiple iterations of the algorithm. The case where the instance of oracular defect varies over iterations has been discussed in Ref. [34]. In our case, we have the following search Hamiltonian

$$H^{dis}_{search} = -|w\rangle\langle w| - |s\rangle\langle s| + \sum_{i=1}^{n} \epsilon_i |i\rangle\langle i|, \tag{3}$$

where ϵ_i-s are the value of static disorder at vertex i and are i.i.d random variables from some probability distribution of mean 0 and standard deviation $\sigma \ll 1$. In fact, the form of the probability distribution is not very important for the results we derive, as long as there is a high probability that $|\epsilon_i| < \sigma$, and also that in a typical instance we have ϵ_i to be of the same order as σ.

The approximate eigenstates and eigenvalues of $H^{\text{dis}}_{\text{search}}$ are calculated in Sec. I of the Supplemental Material, whereas here we summarize the results. Let $|s_{\bar{w}}\rangle$ be the equal superposition of all nodes other than the solution node $|w\rangle$. Then by using degenerate perturbation theory, we find that the approximate ground and first excited states of the system are obtained by diagonalizing the search Hamiltonian projected onto the subspace spanned by $\{|w\rangle, |s_{\bar{w}}\rangle\}$. The Hamiltonian of the effective two level system is

$$H_{red} = \begin{bmatrix} -1+\epsilon_w & -1/\sqrt{n} \\ -1/\sqrt{n} & -1 \end{bmatrix}, \tag{4}$$

which interestingly only depends on the error at the oracle ϵ_w. The gap between the ground state and the first excited state of the perturbed Hamiltonian is

$$\Delta \approx \sqrt{\epsilon_w^2 + 4/n} \tag{5}$$

and the success probability of the algorithm is given by

$$P_w(t) = |\langle w|e^{-iH_{red}t}|s\rangle|^2 \approx \frac{1}{1 + n\epsilon_w^2/4} \sin^2(\frac{\Delta t}{2}), \tag{6}$$

which is plotted in blue in Fig. 1. The maximum success probability is achieved at a time $T = \pi/\Delta$ and since it is lower than 1, we need to repeat the algorithm $1/P_w(T)$ times on average to find the marked node. Hence, Eq. (6) shows that there are two distinct regimes for the average running time

• *Weak disorder* ($\sigma \leq \mathcal{O}(1/\sqrt{n})$): The maximum success probability is constant and the frequency $\Delta = \mathcal{O}(1/\sqrt{n})$. Thus, the algorithm remains optimal.

• *Strong disorder* ($\sigma > \mathcal{O}(1/\sqrt{n})$): The maximum success probability scales as $\mathcal{O}(1/(n\sigma^2))$ and $\Delta = \mathcal{O}(\sigma)$. Thus,

one needs to repeat the algorithm $\sim n\sigma^2$ times on average, to obtain an expected running time of $\mathcal{O}(n\sigma)$.
This implies that a high degree of control in the system is necessary to maintain quantum speed-up. In fact, unless it is possible to decrease the disorder strength σ with the system size, only a constant speed-up is possible with respect to the classical case where search takes $\mathcal{O}(n)$ time.

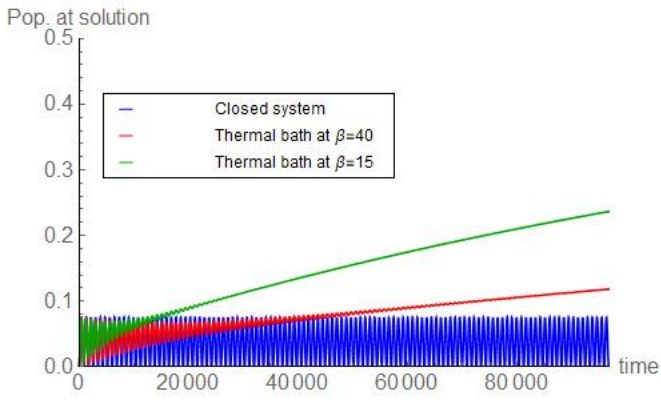

FIG. 1: Comparison of the population at the solution node with time for a complete graph of 1000000 nodes where each node of the graph is affected by diagonal disorder of maximum strength $\sigma = 0.007$. Blue curve indicates the closed system dynamics. The disordered system is coupled to a thermal bath having an ohmic spectral density with a cut-off frequency $\omega_c = 2$ while the system-bath coupling $g = 0.02$. We numerically solve the Bloch-Redfield master equation. The red curve corresponds to the population at the solution when the thermal bath is at an inverse temperature $\beta = 40$ while the green curve corresponds to $\beta = 15$. We find that the amplitude at the solution is always low in the unitary regime implying that the algorithm needs to be repeated several times. On the other hand, interaction with the thermal bath results in amplifying the population at the solution with time without compromising much in the algorithmic running time. Moreover, increasing the temperature of the bath ensures faster relaxation and improves the running time of the algorithm.

Analog quantum search with diagonal disorder in the presence of a thermal bath.— We shall now see how the coupling of the system to a thermal bath can increase the success probability of the algorithm due to thermal relaxation. We will focus our analysis on the strong disorder regime, since it is more realistic to assume that we would not have sufficient control on the system to ensure that the disorder strength σ is less than $1/\sqrt{n}$, given that the dimension of the Hilbert space n increases exponentially with the number of qubits.
By looking at the approximate two level description of H^{dis}_{search} given in Eq. 4 one can see that the transitions from $|s_{\bar{w}}\rangle$ to $|w\rangle$ are suppressed due to the energy mismatch ϵ_w. On coupling the system to a thermal bath, we expect it to evolve to a thermal state which enhances the aforementioned transition due to thermal relaxation. In fact, in the zero temperature regime, we expect the system to relax to the ground state and thus, if the ground state has a large overlap with $|w\rangle$, we obtain a maximum probability of success close to 1, in spite of disorder.
In the strong disorder regime, we obtain that the ground state of H_{red} is approximately $|w\rangle$ only if the random variable $\epsilon_w \ll -1/\sqrt{n}$, which happens with probability of approximately $1/2$ assuming the probability distribution is symmetric around 0. In order to ensure that almost always $|w\rangle$ has a large overlap with the ground state we choose the parameter $\gamma = (1-\sigma)/n$, instead of the value $1/n$ mentioned before and chosen in Ref. [21]. This choice does not change the scaling of the average running time of the search algorithm with disorder, which is still $\mathcal{O}(n\sigma)$ on average and requires $\sim n\sigma^2$ repetitions. The gap between the ground state and the first excited state, as result of this choice of γ becomes $\Delta = \sigma - \epsilon_w + \mathcal{O}(1/(n\sigma))$ and the approximate eigenstates to first order in perturbation theory are

$$|\lambda_1\rangle \approx |w\rangle + \frac{1}{\sqrt{n}(\sigma - \epsilon_w)} |s_{\bar{w}}\rangle \,, \tag{7}$$

$$|\lambda_2\rangle \approx \frac{1}{\sqrt{n}(\sigma - \epsilon_w)} |w\rangle - |s_{\bar{w}}\rangle \,, \tag{8}$$

which are obtained in an analogous way as shown in Sec. I of the Supplemental Material. With this new choice of γ, the overlap of the ground state with the marked node is close to 1 with high probability, as desired.
We consider the following Hamiltonian which describes the interaction of the system with a thermal bath of harmonic oscillators

$$H_I = \sum_{i=1}^{n} \sum_{\alpha} g_{i\alpha}(a_{i\alpha} + a^\dagger_{i\alpha}) |i\rangle \langle i| \,, \tag{9}$$

where $a^\dagger_{i\alpha}$ and $a_{i\alpha}$ are the bath creation and annihilation operators obeying $[a_{i\alpha}, a^\dagger_{j\beta}] = \delta_{i,j}\delta_{\alpha,\beta}$, i.e. we consider that each node $|i\rangle$ of the complete graph is coupled to a bosonic bath, which we assume to be at an inverse temperature β (throughout the article we are working in units where Boltzmann constant $k_B = 1$). Furthermore we assume that the bath temperature is low enough so that the transitions to states higher than the first excited state are negligible. To ensure that this happens we need $\beta \gg \beta^* = \mathcal{O}(\log(n))$.
To describe the evolution of the system's density matrix we first assume that the coupling between each site of the system and the bath is considered to be identical ($g_{i\alpha} = g$, for all i, α) and that g is sufficiently weak so that the system and the bath remain uncorrelated at all

times. Secondly, we consider that the time scale of decay of the bath correlation functions δt is much faster than the relevant time-scales of the system, i.e. the Markov approximation is valid. The condition $g \ll 1/\delta t$ ensures that this is indeed the case. These assumptions lead us to the well known Bloch-Redfield master equation [24, 25, 35]. This equation allows us to resolve system time-scales which, for the weak disorder regime, are of $O(\sqrt{n})$ and thus are important to understand the regimes where the algorithm remains optimal. This analysis is done in Sec. II-IV of the Supplemental material. In the strong disorder regime, as the system time-scales are of $O(1/\sigma) \ll \sqrt{n}$ we are not interested in resolving this time scale and so we can take the secular approximation [25]. The condition $g \ll \sqrt{\sigma/\delta t}$ ensures that both the secular and Markov approximations are valid (see Sec. V of the Supplemental Material).

Let $\rho_{ij} = \langle\lambda_i| \rho |\lambda_j\rangle$ be the density matrix elements of the system, expressed in its eigenbasis. The master equation that describes the time-evolution of the population of ground and first excited states of the system after taking the secular approximation is

$$\dot{\rho}_{kk} = \sum_{l\neq k} W_{kl}\rho_{ll} - \sum_{l\neq k} W_{lk}\rho_{kk}, \tag{10}$$

where $k \in \{1, 2\}$ and $l \in \{1, 2\}$. The transition rates are given by

$$W_{kl} = \begin{cases} 2\pi J(\omega_{kl})\Lambda_{kl}\mathcal{N}(\omega_{kl}), & \omega_{kl} < 0 \\ 2\pi J(\omega_{lk})\Lambda_{kl}\Big(\mathcal{N}(\omega_{lk}) + 1\Big), & \omega_{kl} \geq 0 \end{cases} \tag{11}$$

with $\mathcal{N}(\omega) = 1/(e^{\beta\omega} - 1)$ and $\Lambda_{kl} = \sum_i |c_{ik}c_{il}|^2$. The coefficients c_{ik} are obtained from the basis change $|i\rangle = \sum_k c_{ik} |\lambda_k\rangle$. We consider the spectral density of the bath to be ohmic with an exponential cut-off, i.e. $J(\omega) = \eta g^2 \omega e^{-\omega/\omega_c}$, and that the cut-off frequency ω_c to be a constant larger than the system energy scale, i.e. $\omega_c > 1$. Also, η is a constant normalization factor.

From Eq. (7), we see that the population at the solution is approximately the population of the ground state. Using this and Eq. (10), we obtain

$$P_w(t) \approx \rho_{11}(t) + \mathcal{O}\left(\frac{1}{\sigma\sqrt{n}}\right) \tag{12}$$

$$\approx \frac{1 - e^{-t/T_{rel}}}{1 + e^{-\beta\Delta}} + \mathcal{O}\left(\frac{1}{\sigma\sqrt{n}}\right). \tag{13}$$

The relaxation time is

$$T_{rel} \sim \frac{1}{\Lambda_{12} J(\Delta)} \tanh\left(\frac{\beta\Delta}{2}\right), \tag{14}$$

where Λ_{12} can be calculated from Eqs. (7) and (8) which yield $\Lambda_{12} = \mathcal{O}\left(1/(n\sigma^2)\right)$. We obtain thus a quantum algorithm that is run simply by thermal relaxation and whose running time is given by T_{rel}. The probability of success is given by the ground state population of the Gibbs state

$$P_{suc} = (1 + \exp(-\beta\Delta))^{-1} \tag{15}$$

which is always larger than 1/2. This is an important advantage with respect to the unitary, disordered algorithm since the population at the solution node only increases with time and avoids the need to repeat the algorithm several times.

However, a careful analysis of the relaxation time is needed to ensure that any quantum advantage remains.

Zero temperature ($\beta \to \infty$): When the thermal bath is at zero temperature, i.e. when $\beta \to \infty$, the relaxation time of the system is $T_{rel}(\infty) = \mathcal{O}\left(n\sigma/\eta g^2\right)$.

High temperature ($\beta^* \ll \beta \ll 1/\sigma$): In this regime of temperature, $\tanh(\beta\sigma/2) \approx \beta\sigma/2$. This gives us that the relaxation time, $T_{rel}(\beta) = \mathcal{O}\left(n\sigma^2\beta/\eta g^2\right)$. Thus the ratio,

$$\tau = \frac{T_{rel}(\beta)}{T_{rel}(\infty)} = \beta\sigma \ll 1. \tag{16}$$

This shows that increasing the temperature actually ensures faster relaxation to the thermal state thereby improving the algorithmic running time. This has been plotted in Fig. 1 where we find that relaxation is faster for the thermal bath at $\beta = 15$ (green) as compared to $\beta = 40$ (red). Also observe the distinct difference in the dynamics of the population at solution of these two curves as compared to the unitary scenario (blue). The probability at the solution is considerably higher in the presence of a thermal bath.

In order to analyse the fastest relaxation time we can obtain in this framework, it is crucial note that the vailidity of the secular and Markov approximations implies that we have to restrict the system-bath coupling to a value $g \ll \sqrt{\sigma/\delta t}$. The larger the g the fastest the relaxation, and so the relaxtion time is minimized for $g = \chi\sqrt{\sigma/\delta t}$, where χ is some small constant.

We prove in Secs. VI and VII of Supplemental Material that the bath-correlation timescale δt is $\delta t \sim \omega_c$ at zero temperature and is given by $\delta t \sim \beta$ at finite temperature. This implies, for zero temperature the lower bound for the relaxation time is $T_{rel}(\infty) = \Omega(n)$ which is no better than classical search. For finite temperatures however, we have that $T_{rel}(\beta) = \Omega\left(n\sigma\beta^2\right)$. The relaxation time decreases for higher temperatures but it is necessary to keep $\beta > \mathcal{O}(\log(n))$ for the two-level approximation to be valid. Hence, the fastest relaxation possible in this framework is

$$T_{rel} = \mathcal{O}\left(n\sigma(\log n)^2\right), \tag{17}$$

which matches the running time of the unitary disordered case up to a logarithmic factor.

From our observation it is perhaps tempting to conjecture whether a stronger value of g can improve the relaxation time further and whether one can obtain a running time that is better than $\mathcal{O}(\sqrt{n})$, albeit violating the assumption that the bath is Markovian. To the contrary in Sec. VIII of the Supplemental Material, we use the proof of Ref. [21] to show that the lower bound for any open quantum search algorithm is in fact $\mathcal{O}(\sqrt{n})$. We show in Sec. III of the Supplemental Material, that this lower bound can be obtained in the absence of static disorder, for certain ranges of bath parameters.

Discussion— We have analyzed the robustness of quantum analog search algorithm in the presence of diagonal static disorder and showed that the algorithm loses optimality for a disorder strength $\sigma > \mathcal{O}(1/\sqrt{n})$. In this regime, the success probability decreases with the system size and the algorithm needs to be repeated $n\sigma^2$ times on average, to have a running time of $\mathcal{O}(n\sigma)$.

We have shown that, if this system is coupled to a thermal bath, it is possible to significantly increase the success probability of the algorithm, from $1/(n\sigma^2)$ to a fixed value larger than $1/2$, due to thermal relaxation. Moreover, the algorithmic running time improves with temperature due to faster relaxation. For an appropriate choice of bath parameters, we obtain an algorithm in the open regime whose running time is close to that of the disordered unitary case, with the added advantage that only a constant number of repetitions are needed.

It would be interesting to explore whether a similar effect holds when there is an unkown number of solutions to the search problem. In such a case, the dissipative dynamics could lead to a new quantum algorithm for fixed-point search assisted by the environment, requiring no additional resources [36, 37].

Our results can also be extended to the spatial search algorithm on other graphs [38–40] provided that there is a large gap between the two highest eigenvalues of the adjacency matrix of the graph, as the two level approximation remains valid. Furthermore, it would be worth exploring whether a non-Markovian bath could lead to faster thermal relaxation and whether it is possible to get close to the proven lower bound of $\mathcal{O}(\sqrt{n})$ even in the presence of strong disorder [20, 41].

Finally, our work opens up possibilities to study whether non-engineered open quantum system dynamics can be beneficial for other analog quantum algorithms affected by static errors.

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Supplemental Material

I. ANALOG QUANTUM SEARCH WITH STATIC ERRORS

The analog search algorithm requires an oracle that marks the solution node to an energy that is different from the rest of the nodes. In order for the problem to have a fair comparison to the standard Grover's algorithm in the circuit model, the energy at the marked node is chosen to be -1 [22]. However an imperfect implementation of the oracle might severely affect the algorithmic performance. We define an imperfect oracle as one which "marks" each node of the graph erroneously: each node non-solution node j is marked with an energy ϵ_j, while the solution node, w is marked with an energy $-1+\epsilon_w$ (where each ϵ_w is a random variable). The resultant effect can be perceived as an introduction of static disorder to the nodes of the complete graph. We consider that these errors are systematic i.e. we assume that the value of each ϵ_j does not change over different iterations of the algorithm. We have thus the following search Hamiltonian

$$H^{dis}_{search} = -|w\rangle\langle w| - |s\rangle\langle s| + \sum_{i=1}^{n} \epsilon_i |i\rangle\langle i|, \tag{S1}$$

where ϵ_i is the value of static disorder at vertex i and are i.i.d random variables from some probability distribution of mean 0 and width 2σ. In fact, the form of the probability distribution is not very important for the results we derive, as long as there is a high probability that $-\sigma \leq \epsilon_i \leq \sigma$, and also that in a typical instance we have ϵ_i to be of the same order as σ. We assume that $\sigma \ll 1$ and that one can estimate the value of σ without having access to the

individual ϵ_is. Using perturbation theory we calculate the approximate system eigenstates. To do so, let us rewrite the Hamiltonian in the following form

$$H^{dis}_{search} = \underbrace{-|w\rangle\langle w| - |s_{\bar{w}}\rangle\langle s_{\bar{w}}|}_{H_0} \underbrace{- \frac{1}{\sqrt{n}}(|w\rangle\langle s_{\bar{w}}| + |s_{\bar{w}}\rangle\langle w|) + \sum_{i=1}^{n}\epsilon_i |i\rangle\langle i|}_{V}, \tag{S2}$$

where we have neglected terms of order $O(1/n)$. It is expected that the strength of the perturbation V should be dominated by the disorder when $\sigma \gg 1/\sqrt{n}$, whereas if $\sigma \ll 1/\sqrt{n}$ we expect the algorithm to be unaffected by disorder. In fact we will see that this threshold is very important for the running time of the algorithm.
Since H_0 has two degenerate eigenstates $|w\rangle$ and $|s_{\bar{w}}\rangle$, we apply degenerate perturbation theory to obtain the approximate ground and first excited states of the system. At first order, these are calculated by the diagonalization of the Hamiltonian projected onto this degenerate subspace, which is given by

$$H_{red} = \begin{bmatrix} -1+\epsilon_w & -1/\sqrt{n} \\ -1/\sqrt{n} & -1+\bar{\epsilon} \end{bmatrix}, \tag{S3}$$

where, ϵ_w is the strength of disorder at the solution node $|w\rangle$ and $\bar{\epsilon} = \sum_{i\neq w}\epsilon_i/(n-1)$ is the mean of the disorder at all sites other than the solution. We will neglect the random variable $\bar{\epsilon}$ because it has 0 mean and its fluctuations are of $\mathcal{O}(\sigma/\sqrt{n}) \ll \mathcal{O}(1/\sqrt{n})$.
From Eq. (S3) it is clear that the dynamics is dominated by the value of disorder at the marked vertex. The diagonalization of H_{red} yields the following eigenvectors

$$|\lambda_1^{(1)}\rangle = \frac{1}{K}\left(\frac{1}{\sqrt{n}}|w\rangle + \left(\Delta - \frac{\epsilon_w}{2}\right)|s_{\bar{w}}\rangle\right) \tag{S4}$$

$$|\lambda_2^{(1)}\rangle = \frac{1}{K}\left(\left(\frac{\epsilon_w}{2} - \Delta\right)|w\rangle + \frac{1}{\sqrt{n}}|s_{\bar{w}}\rangle\right), \tag{S5}$$

where $K = \sqrt{(\frac{\epsilon_w}{2} - \Delta)^2 + 1/n}$ is the normalization factor. The corresponding eigenvalues are

$$\lambda_1^{(1)} = -1 + \epsilon_w/2 - \Delta \tag{S6}$$

$$\lambda_2^{(1)} = -1 + \epsilon_w/2 + \Delta \tag{S7}$$

where the gap Δ is given by $\Delta = \lambda_2 - \lambda_1 = \sqrt{\epsilon_w^2 + 4/n}$. The success probability of the algorithm, also calculated at first order in perturbation theory, is given by

$$P_w^{(1)}(t) = |\langle w|\exp^{-iH_G t}|s\rangle|^2 \approx \frac{1}{1+n\epsilon_w^2/4}\sin^2(\frac{\Delta t}{2}). \tag{S8}$$

The probability $P_w^{(1)}(t)$ is maximum at $T' = \pi/\Delta$, $P_w^{(1)}(T') = 1/(1+n\epsilon_w^2/4)$ and hence the algorithm needs to be repeated $1/P_w^{(1)}(T')$ times on average in order to find the marked vertex. This gives the average running time as

$$T_{dis} = \frac{\pi\sqrt{n}}{2}\sqrt{1 + \frac{n\epsilon_w^2}{4}}, \tag{S9}$$

where we assumed that ϵ_w takes the same value if one repeats the algorithm using the same system (it is a systematic error). We have thus two regimes of disorder:

Weak diagonal disorder regime: As long as $\sigma \leq \mathcal{O}(1/\sqrt{n})$, we have that $n\epsilon_w^2 < 1$. Thus, in this regime the algorithm keeps an optimal running time of $\mathcal{O}(\sqrt{n})$ as after this time the probability of observing the solution state is a constant.

Strong diagonal disorder regime: However beyond this threshold of σ, i.e. when $n\sigma^2 \gg 1$, we expect that with high probability $n\epsilon_w^2 \gg 1$ and thus the gap between the ground state and the first excited state is $\Delta \approx |\epsilon_w|/2 \leq \sigma/2$. Also from Eq. (S8) we find that after a time of $T_1 = \pi/\epsilon_w = \mathcal{O}(\pi/\sigma)$, the probability of observing the solution is $\mathcal{O}(1/n\sigma^2)$. Thus the algorithm needs to be repeated $\mathcal{O}(n\sigma^2)$ times to obtain an average running time

of $T = \mathcal{O}(n\sigma)$. If we assume that σ depends on n as $\sigma = n^{-\alpha}$, the algorithm is sub-optimal for $\alpha < 1/2$. The ground state of the Hamiltonian (not normalized) is given by

$$|\lambda_1^{(1)}\rangle \approx \begin{cases} -|w\rangle + \frac{1}{|\epsilon_w|\sqrt{n}}|s_{\bar{w}}\rangle + \mathcal{O}(\frac{1}{n\epsilon_w^2}), \text{ if } \epsilon_w < 0 \\ -\frac{1}{|\epsilon_w|\sqrt{n}}|w\rangle + |s_{\bar{w}}\rangle + \mathcal{O}(\frac{1}{n\epsilon_w^2}), \text{ if } \epsilon_w > 0 \end{cases} \tag{S10}$$

and thus it has a large overlap either with $|w\rangle$ or with $|s_w\rangle$ depending on the sign of the random variable ϵ_w. As explained in the main text we can ensure that the ground state has always a larger overlap with the solution by shifting the parameter γ.

At this point, it is also important to understand the order of magnitude of the terms we have neglected in perturbation theory. The magnitude of the second order corrections to the eigenvalues λ_1 and λ_2 is of $\mathcal{O}(|V|^2)$. This means that we expect that Eq. (S8) is valid for a timescale $t \ll 1/|V|^2$, which is sufficient for the discussion of the running time that we have done previously. Furthermore, it is possible to show that the terms we have neglected in the probability due to second order corrections to the eigenstates are of $\mathcal{O}(1/n)$.

II. BLOCH-REDFIELD MASTER EQUATION FOR A TWO LEVEL SYSTEM

The interaction of a quantum system with a thermal bath induces relaxation dynamics which eventually leads the system to a thermal state. We want to calculate the population at the solution state $|w\rangle$ with time. The analog search Hamiltonian can be approximated by a two level system as long as the temperature of the bath is less than the gap between the first excited state and the rest of the energy levels. As seen previously, this is also true when the algorithm is affected by static disorder at the nodes of the graph. Throughout our analysis we shall assume that the bath temperature is such that the system can be well approximated by a two level system.
Consider a system Hamiltonian which can be approximated as a two-level system with Δ being the gap between the ground state and the first excited state. We assume that the system interacts with a thermal bath that whose Hamiltonian is given by

$$H_R = \sum_{i=1}^{n}\sum_{\alpha}\omega_\alpha a_{i\alpha}^\dagger a_{i\alpha}, \tag{S11}$$

with $[a_{i\alpha}, a_{j\beta}^\dagger] = \delta_{i,j}\delta_{\alpha,\beta}$. Furthermore, we will consider the interaction Hamiltonian given by

$$H_I = \sum_{i=1}^{n}\sum_{\alpha} g_{i\alpha}(a_{i\alpha} + a_{i\alpha}^\dagger)\,|i\rangle\langle i|\,, \tag{S12}$$

i.e. each node of the graph is coupled to an independent bosonic bath, which we assume to be in a thermal state at temperature $1/\beta$ (throughout the article we are working in units where the Boltzmann constant $k_B = 1$). Each site $|i\rangle$, rewritten in the eigenbasis of the system is expressed as $|i\rangle = \sum_k c_{ik}|\lambda_k\rangle$. Let $\rho_{ij} = \langle\lambda_i|\,\rho\,|\lambda_j\rangle$, where ρ denotes the density matrix of the system. We are interested in calculating the population of the solution with time which is given by

$$P_w(t) = \rho_{11}(t)|\langle w|\lambda_1\rangle|^2 + \rho_{22}(t)|\langle w|\lambda_2\rangle|^2 + \rho_{12}(t)\langle w|\lambda_1\rangle\langle\lambda_2|w\rangle + \rho_{21}(t)\langle w|\lambda_2\rangle\langle\lambda_1|w\rangle\,, \tag{S13}$$

where each $\rho_{ij}(t)$ is given by solution to the Bloch-Redfield Master equation . That is

$$\dot{\rho_{ab}} = -i\omega_{ab}\rho_{ab} + \sum_{abcd} R_{abcd}\rho_{cd}(t), \tag{S14}$$

where $\omega_{ij} = \lambda_i - \lambda_j$. For a two level system, $\{a, b, c, d\} \in \{1, 2\}$ and

$$R_{abcd} = -\frac{1}{2}\sum_j \Big\{\delta_{bd}\sum_x A_{ax}^j A_{xc}^j S_j(\omega_{cx}) - A_{ac}^j A_{db}^j S_j(\omega_{ca}) + \delta_{ac}\sum_x A_{dx}^j A_{xb}^j S_j(\omega_{dx}) - A_{ac}^j A_{db}^j S_j(\omega_{db})\Big\}, \tag{S15}$$

such that $A_{xy}^j = c_{jx}c_{jy}^*$. Also,

$$S_i(\omega_{kl}) = \begin{cases} J(\omega_{kl})\mathcal{N}(\omega_{kl}), & \omega_{kl} < 0 \\ J(\omega_{lk})\Big(\mathcal{N}(\omega_{lk}) + 1\Big), & \omega_{kl} \geq 0 \end{cases}, \tag{S16}$$

with $\mathcal{N}(\omega) = 1/(e^{\beta\omega} - 1)$ and $J(\omega)$ being the spectral density of the bath given by

$$J(\omega) = g^2 \sum_{\alpha} \delta(\omega - \omega_\alpha), \tag{S17}$$

where it is assumed that the coupling between each site of the system and the bath is identical ($g_{i\alpha} = g,$ for all i, α) and sufficiently weak so that the Markov approximation is valid. More precisely, the Markov approximation implies that the time scale of decay of the bath correlation functions, δt is much faster than the relevant time-scales of the system. We show in Sec. V that choosing $g \ll 1/\delta t$ ensures that the Markov approximation is valid. So in our analysis, we fix a value of g so that the Markov approximation is valid. As the nodes the graph are coupled to a set of independent harmonic oscillators, each having the same spectral density and so $S_j(\omega_{xy})$ is the same for all j. We drop this subscript henceforth.
By expressing the two state system density matrix as

$$\rho = \frac{1}{2}\left(I + \vec{n}.\vec{\sigma}\right), \tag{S18}$$

where $\vec{n} = (\rho_x, \rho_y, \rho_z)$ is a vector with real entries and σ_j's are the Pauli matrices with $j \in \{x, y, z\}$. In the Pauli basis, the Bloch-Redfield master equation simplifies to the following set of differential equations:

$$\dot{\rho_x} = -\omega_{12}\rho_y + S(\omega_{12})O_2\rho_z - \frac{S(0)}{2}O_3\rho_x \tag{S19}$$

$$\dot{\rho_y} = \omega_{12}\rho_x - \left\{\frac{1}{2}S(0)O_3 + O_1\left(S(\omega_{12}) + S(\omega_{21})\right)\right\}\rho_y \tag{S20}$$

$$\dot{\rho_z} = S(0)O_2\rho_x - O_1\left(S(\omega_{12}) + S(\omega_{21})\right)\rho_z + O_1\left(S(\omega_{21}) - S(\omega_{12})\right), \tag{S21}$$

where we have that

$$O_1 = \sum_i (A^i_{12})^2 \tag{S22}$$

$$O_2 = \sum_i A^i_{12}(A^i_{11} - A^i_{22}) \tag{S23}$$

$$O_3 = \sum_i (A^i_{11} - A^i_{22})^2. \tag{S24}$$

Throughout the article, we assume that the spectral density of the bath is ohmic with an exponential cut-off, i.e.

$$J(\omega) = \eta g^2 \omega e^{-\omega/\omega_c}, \tag{S25}$$

where ω_c is the cut-off frequency of the bath and η is a constant normalization factor. We fix the cut-off frequency ω_c to be a constant greater than one. For an ohmic bath, $S(0) = \lim_{\omega\to 0^-} S(\omega) = \lim_{\omega\to 0^+} S(\omega) = \eta g^2/\beta$. We shall use this general form of Bloch-Redfield master equation for analyzing how thermal relaxation can assist the analog search algorithm, even when static errors affect the algorithm.

III. ANALOG QUANTUM SEARCH IN THE PRESENCE OF A THERMAL BATH

For the analog search algorithm, we have that the gap between the ground state and the first excited state, $\omega_{21} = \Delta = 2/\sqrt{n}$ and we have that $\langle w|\lambda_1\rangle = \langle w|\lambda_2\rangle = 1/\sqrt{2}$ and hence

$$P_w(t) = \frac{1}{2}\left(1 + \rho_x(t)\right), \tag{S26}$$

where we used the fact that $\rho_{11}(t) + \rho_{22}(t) = 1$ and that $2Re[\rho_{12}(t)] = \rho_x(t)$.
To obtain the master equation corresponding to $\rho_x(t)$ observe that $A^j_{11} = A^j_{22}$. This implies that O_2 and O_3 in Eq. (S23) and Eq. (S24) are 0. Furthermore, $A^i_{21} = A^i_{12} = A_i$. This simplifies the Bloch-Redfield Master equation

considerably as ρ_z is decoupled from ρ_x and ρ_y. Thus to obtain the population of the solution state with time we have to solve the following differential equations:

$$\dot{\rho_x} = \Delta\rho_y \tag{S27}$$
$$\dot{\rho_y} = -\Delta\rho_x - 2\Gamma\rho_y, \tag{S28}$$

where,

$$\Gamma = \frac{1}{2}\sum_i A_i^2\left(S(\Delta) + S(-\Delta)\right) \tag{S29}$$
$$= \frac{(\sum_i A_i^2)J(\Delta)}{2\tanh(\beta\Delta/2)} \tag{S30}$$
$$= \frac{J(\Delta)}{8\tanh(\beta\Delta/2)} \qquad [\ \sum_i A_i^2 = 1/4 + \mathcal{O}(1/n)\ ]. \tag{S31}$$

The solution to the Bloch-Redfield Master Equation is

$$\rho_x(t) = e^{-\Gamma t}\Big[\frac{\Gamma}{\sqrt{\Gamma^2-\Delta^2}}\Big\{\frac{e^{-\left(\sqrt{\Gamma^2-\Delta^2}\right)t} - e^{\left(\sqrt{\Gamma^2-\Delta^2}\right)t}}{2}\Big\} - \Big\{\frac{e^{-\left(\sqrt{\Gamma^2-\Delta^2}\right)t} + e^{\left(\sqrt{\Gamma^2-\Delta^2}\right)t}}{2}\Big\}\Big]. \tag{S32}$$

From Eq. (S32) we find that there arises two distinct cases that determine the nature of relaxation dynamics:

(i) Underdamped relaxation to the steady state ($\Gamma < \Delta$): When $\sqrt{\Gamma^2-\Delta^2}$ is imaginary, we have that

$$P_w(t) = \frac{1}{2}\left(1 - e^{-\Gamma t}\left\{\frac{\Gamma}{\Delta}\sin\left[\left(\sqrt{\Delta^2-\Gamma^2}\right)t\right] + \cos\left[\left(\sqrt{\Delta^2-\Gamma^2}\right)t\right] + \mathcal{O}(\Gamma^2/\Delta^2)\right\}\right), \tag{S33}$$

where ,

$$\Gamma = \mathcal{O}\left(\frac{\eta g^2\Delta}{\tanh(\beta\Delta/2)}\right), \tag{S34}$$

We find that after a time of $T_{rel}(\beta) = \mathcal{O}(\Delta^{-1})$, there is constant population at the solution. Note that the steady state is reached after a time $\mathcal{O}(1/\Gamma)$ and $P_w(\infty) = 1/2$. Note that larger the temperature, faster is the relaxation rate Γ but the running time of the algorithm is still $\mathcal{O}(1/\Delta)$.

(ii) Over-damped relaxation to the steady state ($\Gamma > \Delta$): In this case $\sqrt{\Gamma^2-\Delta^2}$ is real. Hence

$$P_w(t) = \frac{1}{2}(1 - e^{-t\Delta^2/\Gamma}) + \mathcal{O}(\Delta^2/\Gamma^2). \tag{S35}$$

Thus after a time $T = \mathcal{O}(\Gamma/\Delta^2) = \mathcal{O}(n\Gamma)$, the system reaches a steady state and the population of the solution is constant. Unlike the underdamped case, increasing the temperature makes the relaxation slower.
For a given system-environment coupling strength g, the parameter that determines whether we are in case (i) or (ii) is the temperature of the bath. In particular, we consider two regimes of temperature:

Zero temperature ($\beta \to \infty$): When the thermal bath is at near zero temperature, the bath correlation function decays at a time scale $\delta t = \mathcal{O}\left(1/\omega_c\right)$. In this case we have that

$$\Gamma = \mathcal{O}\left(\eta g^2\Delta\right) \tag{S36}$$
$$= \mathcal{O}\left(\frac{\eta g^2}{\sqrt{n}}\right). \tag{S37}$$

Since $g \ll 1$, we are always in the underdamped regime when the thermal bath is at zero temperature and the population of the solution state is given by Eq. (S33). So after a time $T_{rel}(\infty) = \mathcal{O}(\sqrt{n})$, the probability of being at the solution is a constant which gives the optimal scaling of the running time of the analog search algorithm. Moreover, the population of the solution can only increase for times $t > T$ and hence the algorithm exhibits a fixed

point behavior. This implies that, unlike in the unitary case, one need not measure at the optimal time in order to obtain the solution state with high probability as the steady state of dissipative analog quantum search has a high overlap with the solution state.

High temperature regime ($\mathcal{O}(\log(n)) \ll \beta \ll 1/\Delta$): In this regime, we consider the scenario where the temperature of the bath is greater than Δ, but sufficiently low ($\beta >> \log(n)$) so to ensure that the two level approximation is valid. Note that in this scenario, $\tanh(\beta\Delta/2) \approx \beta\Delta/2$. Also the thermal time-scale is less than the time-scale ($t/\beta >> 1$). As discussed previously and shown in Sec. VII, the correlation time of the thermal bath, $\delta t = \mathcal{O}(\beta)$. The rate Γ is

$$\Gamma = \frac{\eta g^2 \Delta e^{-\Delta/\omega_c}}{8 \tanh(\beta\Delta/2)} \tag{S38}$$

$$= \mathcal{O}\left(\frac{\eta g^2}{\beta}\right). \tag{S39}$$

In this case, increasing the temperature, gradually pushes the dynamics to the overdamped regime as soon as $\Gamma > 1/\sqrt{n}$. In the overdamped regime, the relaxation time

$$T_{rel}(\beta) = \mathcal{O}\left(\frac{\eta g^2 n}{\beta}\right), \tag{S40}$$

is slower with increase in temperature.
Note that whenever, $g^2/\beta \leq \mathcal{O}(1/\sqrt{n})$, we are in the underdamped regime and the algorithmic running time is optimal. Otherwise, running time worsens with increase in temperature. In fact this is clearly exhibited in the ratio between the relaxation times at zero and non-zero temperature. That is,

$$\tau = \frac{T_{rel}\,(\beta)}{T_{rel}\,(\infty)} = \mathcal{O}\left(\frac{\sqrt{n} g^2}{\beta}\right), \tag{S41}$$

increases with increase in temperature.

IV. ANALOG QUANTUM SEARCH WITH DIAGONAL DISORDER IN THE PRESENCE OF A THERMAL BATH

The presence of static disorder changes the ground state and the first excited state of the algorithm. In fact now $|\langle\lambda_1|w\rangle| \neq |\langle\lambda_2|w\rangle|$. In the presence of a thermal bath, we require that the ground state of the system Hamiltonian has a higher overlap with the solution state. This is because the relaxation dynamics due to the thermal bath would increase the population at the solution. In order to ensure that the ground state of the search Hamiltonian has a higher overlap with the solution state $|w\rangle$ we need $\langle w|H_{red}|w\rangle < \langle s_{\bar{w}}|H_{red}|s_{\bar{w}}\rangle$, which can be achieved by an appropriate choice of the parameter γ. A possible choice is $\gamma = (1-\sigma)/n$. The gap between the ground state and the first excited state, as result of this choice of γ is

$$\Delta = \sigma - \epsilon_w + \mathcal{O}(1/\sqrt{n}). \tag{S42}$$

As explained in Sec. I, there are two regimes of static disorder and for each of which the analysis for the relaxation of the system is going to differ.

Weak diagonal disorder

In this regime, i.e. when the strength of disorder, $\sigma < \mathcal{O}(1/\sqrt{n})$, the analog search algorithm remains robust to this error and the optimal running time is maintained. Note that the ground state and the first excited state have a constant overlap with the solution state. As mentioned previously, the new choice of γ ensures that the ground state has a higher overlap with the solution state as compared to the first excited state. Also the gap between the ground state and the first excited state $\Delta \sim \mathcal{O}(1/\sqrt{n})$.

In the presence of the thermal bath, the behavior of the analog search algorithm is similar to the scenario where there was no disorder. However, in this regime $A^i_{11} \neq A^i_{22}$ and so O_2 and O_3 are non-zero. The Bloch-Refield equations in Eq. (S19)-(S21) are written as

$$\begin{bmatrix}\dot{\rho_x}\\ \dot{\rho_y}\\ \dot{\rho_z}\end{bmatrix} = \underbrace{\begin{bmatrix}-\frac{1}{2}S(0)O_3 & -\omega_{12} & S(\omega_{12})O_2\\ \omega_{12} & -\frac{1}{2}S(0)O_3 - 2\Gamma & 0\\ S(0)O_2 & 0 & -2\Gamma\end{bmatrix}}_{M}\begin{bmatrix}\rho_x\\ \rho_y\\ \rho_z\end{bmatrix} + \begin{bmatrix}0\\ 0\\ O_1(S(\omega_{21}) - S(\omega_{12}))\end{bmatrix}, \tag{S43}$$

where $\omega_{21} = \Delta$ and $\Gamma = O_1 J(\Delta)\coth(\beta\Delta/2)/2$. The quantities O_1, O_2 and O_3 are $\mathcal{O}(1)$.

Zero temperature ($\beta \to \infty$): At zero temperature, $S(0) = g^2/\beta = 0$ which simplifies the master equation. We find that $\Gamma = \mathcal{O}\left(\eta g^2/\sqrt{n}\right)$ and the relaxation time is $\mathcal{O}(\sqrt{n})$ which has the same scaling as the case where no static error is present (See Eq. (S37)).

High temperature ($\mathcal{O}(\log(n)) \ll \beta \ll 1/\Delta$): In this regime, no simplification to the master equation is possible and we resort to numerical simulations. Intuitively, one would expect that the behavior of the algorithm is similar to the scenario where there was no disorder. We numerically verify that this is indeed the case and plot the population at the solution with time at high temperature in Fig. S1. We find that the system relaxes to the steady state which is expected to be a statistical mixture between the solution state $|w\rangle$ and the equal superposition of the rest of the nodes ($|s_{\bar{w}}\rangle$).

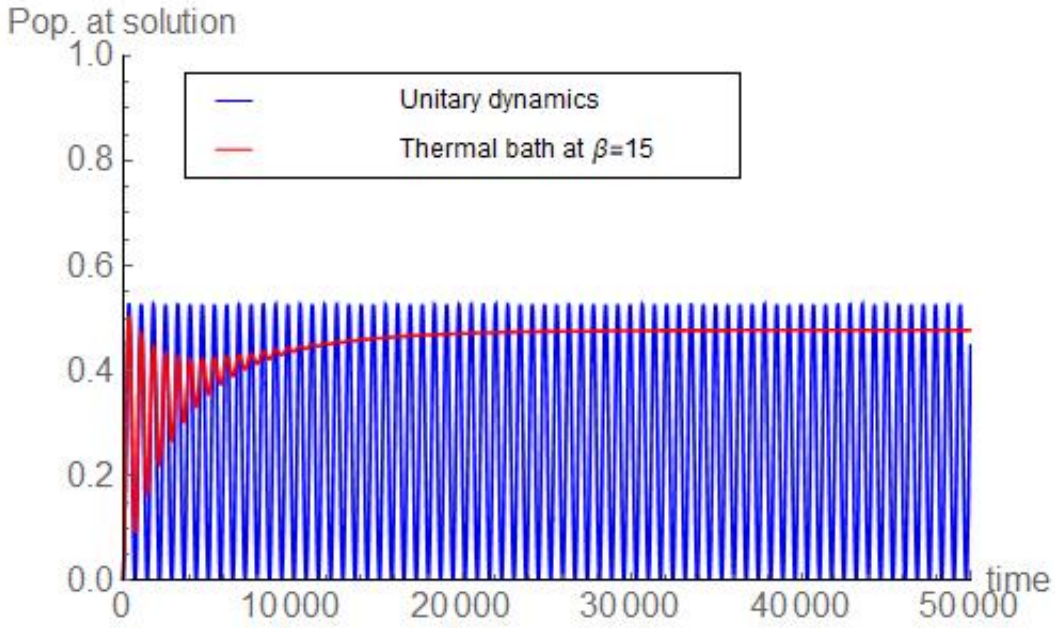

FIG. S1: Comparison of population at the solution node with time for a complete graph of 100000 nodes where each node of the graph is affected by weak diagonal disorder of standard deviation $\sigma = 0.006$ in the unitary regime and in the presence of a thermal bath having a cut-off frequency of $\omega_c = 2$ and system-bath coupling $g = 0.04$. Blue curve indicates the population in the unitary scenario, i.e. in the absence of a thermal bath. The red curve shows the population at the solution in the presence of a thermal bath at inverse temperature, $\beta = 15$. The steady state of the thermal bath has an overlap of close to 1/2 with the solution state.

Strong diagonal disorder

When the strength of disorder, $\sigma > 1/\sqrt{n}$, the analog search algorithm loses its optimality. One observes that one needs to measure after a time $T = \mathcal{O}(\pi/\sigma)$, to find the solution with probability $\mathcal{O}(1/(n\sigma^2))$. Further this probability

is amplified by repeating the algorithm $\mathcal{O}(1/(n\sigma^2))$ times, thereby obtaining an expected running time of $T = \mathcal{O}(n\sigma)$. We show that the introduction of a thermal bath can amplify the amplitude of the solution node with at most a constant overhead in running time even at constant temperatures. In fact, increasing the temperature ensures faster relaxation to the steady state which has a high overlap with the solution state. Moreover, the resultant dissipative dynamics ensures that the population at the solution node only increases with time thereby circumventing the need to repeat the algorithm several times as in the unitary case.

Choosing $\gamma = (1-\sigma)/n$ yields the approximate eigenstates (not normalized) as

$$|\lambda_1\rangle \approx |w\rangle + \frac{1}{\sqrt{n}(\sigma - \epsilon_w)} |s_{\bar{w}}\rangle \,, \tag{S44}$$

$$|\lambda_2\rangle \approx \frac{1}{\sqrt{n}(\sigma - \epsilon_w)} |w\rangle - |s_{\bar{w}}\rangle \,, \tag{S45}$$

by neglecting terms higher order terms that will remain small as long as $\sigma \ll 1$, where the gap $\lambda_2 - \lambda_1$ is

$$\Delta \approx \sigma - \epsilon_w = \mathcal{O}(\sigma). \tag{S46}$$

Thus there is a gap of $1 - \Delta = 1 - \mathcal{O}(\sigma)$ between the first excited state and the rest of the spectrum which is a constant as long as $\sigma \ll 1$. This enables us to approximate our system as a two level system for low temperatures, i.e. $\beta \gg \mathcal{O}(\log(n))$.

The transition from the weak disorder regime to the strong disorder regime is interesting. Observe that as the strength of disorder increases, the component of $\rho_z(t)$ increases in the population of the solution $P_w(t)$. Also note that from Eq. (S21), as temperature of the thermal bath increases, $S(0)$ decreases and so $\rho_x(t)$ is nearly decoupled from $\rho_z(t)$. Thus the population at the solution depends approximately only on $\rho_z(t)$, i.e. on the population of the ground state and the first excited state. In fact observe that the population at the solution, given by Eq. (S13) is now

$$P_w(t) \approx \rho_{11}(t) + \mathcal{O}\left(\frac{1}{\sigma\sqrt{n}}\right), \tag{S47}$$

which implies that the population of the solution is determined by the population of the ground state for $1/\sqrt{n} < \sigma < 1$. The general Bloch-Redfield equation could be simplified in the case of the analog search algorithm in the absence of disorder because of the fact that $A^i_{11} = A^i_{22}$. This is no longer the case in the scenario where the algorithm is affected by disorder. However, in this regime of static disorder, we can coarse grain the time-scale of the relaxation of the system which simplifies the Bloch-Redifield equation considerably. Note that the Bloch-Redfield equation already assumes a coarse graining in the time-scale of the system owing to the Markov approximation. Furthermore the time-scales that we are interested in ($\sim \sqrt{n}$) is significantly greater than the gap $\Delta = 1/\sigma$, we can take the so-called secular approximation which implies an additional course graining in the relaxation dynamics of the system. In general, if g is the strength of coupling between the system and the bath and δt is the width of the correlation function of the bath, the typical relaxation time-scale of the system is $\sim 1/(g^2\delta t)$ and for the secular approximation to hold this has to be greater than $1/\Delta$. Thus, we fix a g that respects both the secular and the Markov approximation. Whenever $\beta\sigma \ll 1$, the choice of g that respects the secular approximation, also respects the Markov approximation. For further details refer to the Sec. VII.

Henceforth, in this section we shall assume that g is such that in addition to the Markov approximation, the secular approximation also holds. Taking the secular approximation ensures that in the Bloch-Redfield equation, the diagonal terms of the density matrix never couples with the off-diagonal terms (Lindblad form). Since from Eq. (S47), we find that the population of the ground state determines the population of the solution, we have the master equation of the dynamics of the population of the ground state and the first excited state.

$$\dot{\rho}_{kk} = \sum_{l \neq k} W_{kl}\rho_{ll} - \sum_{l \neq k} W_{lk}\rho_{kk}, \tag{S48}$$

where $k \in \{1, 2\}$ and $l \in \{1, 2, \ldots, n\}$. The transition rates are given by

$$W_{kl} = \begin{cases} 2\pi J(\omega_{kl})\Lambda_{kl}\mathcal{N}(\omega_{kl}), & \omega_{kl} < 0 \\ 2\pi J(\omega_{lk})\Lambda_{kl}\Big(\mathcal{N}(\omega_{lk}) + 1\Big), & \omega_{kl} \geq 0 \end{cases} \tag{S49}$$

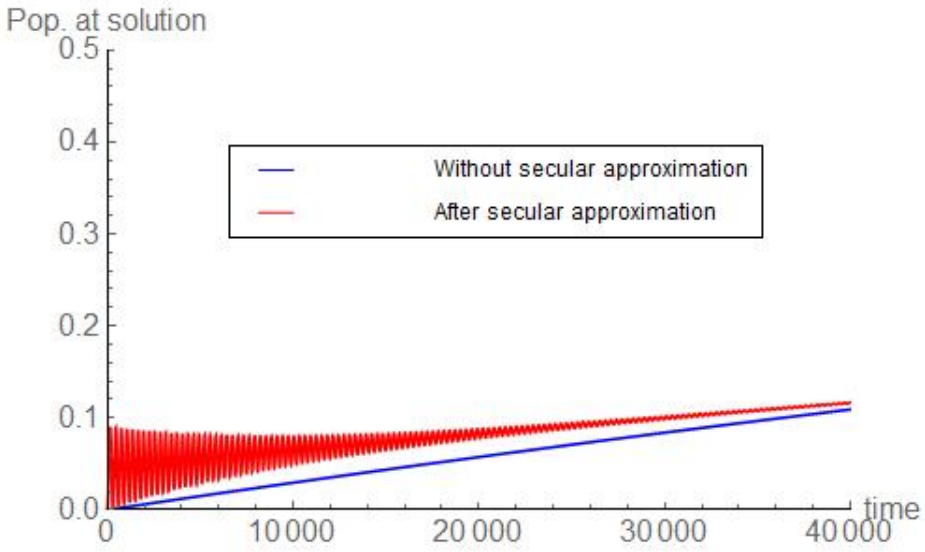

FIG. S2: Plot of the population at the solution node with time for the quantum spatial search algorithm on a complete graph with $n = 100000$ nodes affected by string diagonal disorder of strength $\sigma = 0.02$ in the presence of a thermal bath having an ohmic spectral density with cut off frequency $\omega_c = 2$. The bath is at an inverse temperature $\beta = 20$ and the system bath coupling strength $g = 0.01$. The red curve corresponds to the result obtained by numerically solving the Bloch-Redfield master equation without taking the secular approximation while the blue curve corresponds to our theoretical prediction which takes a secular approximation to Bloch-Redfield master equation.

such that $\Lambda_{kl} = \sum_i |c_{ik}c_{il}|^2$. Solving the differential equation (S48) we obtain,

$$\rho_{11} = \frac{W_{12}}{W_{12}+W_{21}}\left(1 - e^{-(W_{12}+W_{21})t}\right), \tag{S50}$$

$$= \frac{1}{1+e^{-\beta\Delta}}\left(1 - e^{-t/T_{rel}}\right) + \frac{e^{-t/T_{rel}}}{n} \tag{S51}$$

with the relaxation time given by

$$T_{rel} = \frac{1}{W_{12}+W_{21}}. \tag{S52}$$

On substituting the appropriate terms we obtain

$$T_{rel} \sim \frac{1}{\Lambda_{12}J(\Delta)} \tanh\left(\frac{\beta\Delta}{2}\right). \tag{S53}$$

In Fig. S2 we plot the population of the solution node with time in the presence of a thermal bath at non-zero temperature in the scenario where secular approximation is not considered (red curve) as compared to the theoretical prediction in Eq. (S47) (blue curve) where secular approximation has been considered. The plot confirms that the secular approximation is a reasonable approximation to consider.

V. VALIDITY OF THE MARKOV AND SECULAR APPROXIMATIONS

Let H_S represent the Hamiltonian of the system while H_E be the Hamiltonian of the environment. Consider the following interaction Hamiltonian

$$V_I = g\sum_i Q_i F_i, \tag{S54}$$

where Q_i are the system operators and F_i are the bath operators. So in the interaction picture let

$$V_I = g \sum_i Q(t)_i F(t)_i, \tag{S55}$$

where $Q(t)_i$ and $F(t)_i$ are the previously defined system and bath operators represented in the interaction picture. Thus we obtain that after tracing out the environment degrees of freedom, the evolution reduced density matrix of the system is given by the Bloch-Redfield master equation which is of the following form:

$$\frac{d\rho_S}{dt} = \left(\mathcal{L}_{unitary} + \mathcal{L}_{diss}\right) \rho_S(t), \tag{S56}$$

where $\mathcal{L}_{unitary}$ is the super-operator corresponding to purely unitary dynamics while $\mathcal{L}_{diss}$ corresponds to the super-operator corresponding to purely dissipative dynamics.
The Born approximation is respected as long as we are in the weak coupling regime, i.e. $||\mathcal{L}_{relax}|| << ||\mathcal{L}_{unitary}|| = 1$. Note that in the interaction picture, the dynamics of the reduced density matrix of the system (upto $\mathcal{O}(g^2)$) is given by

$$\frac{d\rho_{SI}}{dt} = g^2 \sum_{ij} \int_0^\infty dt' \; Q(t')_i \rho(t')_{SI} Q(t')_j F_{ij}(t') + \text{ other similar terms} \tag{S57}$$

where $F_{ij}(t)$ is the bath correlation function. The bath correlation function decays after bath correlation time-scale defined in the article as δt. Thus the term inside the integral, i.e. $||\mathcal{L}_{diss}|| = \mathcal{O}(g^2 \delta t)$. Now for the Markov approximation to be valid we require that the bath correlation decays faster than the typical time scale of relaxation of the system. This implies that

$$\delta t << \frac{1}{g^2 \delta t} \tag{S58}$$

$$\implies g << \frac{1}{\delta t}. \tag{S59}$$

Also in the main text we find that there arise scenarios (analog search affected by strong diagonal disorder) where secular approximation is to be respected. This means that the typical time scale of the system should be greater than the relevant gaps. In this scenario the gap between the ground and first excited state $\sim \mathcal{O}(\Delta)$. Thus

$$\frac{1}{\Delta} << \frac{1}{g^2 \delta t} \tag{S60}$$

$$\implies g << \sqrt{\frac{\Delta}{\delta t}}. \tag{S61}$$

Then the value of the coupling strength $g < min\{1/\delta t, \sqrt{\Delta/\delta t}\}$.

VI. CORRELATION FUNCTION OF AN OHMIC BATH WITH AN EXPONENTIAL CUTOFF AT ZERO TEMPERATURE

The spectral density of this bath is given by

$$J(\omega) = \eta g^2 \omega e^{-\omega/\omega_c}, \tag{S62}$$

where ω_c is the bath cut off frequency and if $0 < d < 1$, the bath is sub-ohmic, for $d = 1$, the bath is ohmic while for $d > 1$, the bath is super-ohmic. On the other hand, the bath correlation function is given by

$$F_{ii}(t) = \int_0^\infty J(\omega) \Big(\coth(\beta\omega/2) \cos(\omega t) - i \sin(\omega t) \Big) d\omega. \tag{S63}$$

For the Markovian approximation to be valid, the width of the correlation function should decay much faster than the relevant timescales of the system.

At zero temperature,

$$F_{ii}(t) = \eta g^2 \int_0^\infty \omega e^{-\omega/\omega_c} e^{-i\omega t} d\omega \tag{S64}$$

$$= \eta g^2 \int_0^\infty \omega e^{-\omega(\frac{1}{\omega_c}+it)} d\omega \tag{S65}$$

$$= \frac{\eta g^2 \omega_c^2}{(1+it\omega_c)^2} \int_0^\infty q e^{-q} dq \tag{S66}$$

$$\Big[\text{Considering } q = \omega\Big(\frac{1}{\omega_c} + it\Big)\Big] \tag{S67}$$

$$= \frac{\eta g^2 \omega_c^2}{(1+it\omega_c)^2}. \tag{S68}$$

So the width of $F_{ii}(t)$ is $\delta t = \mathcal{O}(1/\omega_c)$.

VII. CORRELATION FUNCTION OF AN OHMIC BATH WITH AN EXPONENTIAL CUTOFF AT NON-ZERO TEMPERATURES

We consider baths with an ohmic spectral density as in Eq. (S62). Considering the bath correlation function defined in Eq. (S63) we have that

$$F_{ii}(t) = \int_0^\infty J(\omega)\Big\{\Big[\frac{1+e^{-\beta\omega}}{1-e^{-\beta\omega}}\Big]\Big(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\Big) - \Big(\frac{e^{i\omega t}-e^{-i\omega t}}{2}\Big)\Big\} d\omega \tag{S69}$$

$$= \int_0^\infty \frac{J(\omega)}{2(1-e^{-\beta\omega})}\Big(e^{-i\omega t} + e^{i\omega t - \beta\omega}\Big) d\omega \tag{S70}$$

$$= \omega_c^{1-d}\Big[\underbrace{\int_0^\infty \frac{\omega^d}{1-e^{-\beta\omega}} e^{-\omega(it+1/\omega_c)} d\omega}_{\mathcal{I}_1} + \underbrace{\int_0^\infty \frac{\omega^d}{1-e^{-\beta\omega}} e^{-\omega(-it+\beta+1/\omega_c)} d\omega}_{\mathcal{I}_2}\Big]. \tag{S71}$$

First we consider the integral $\mathcal{I}_1$. We have

$$\mathcal{I}_1 = \int_0^\infty \frac{\omega^d}{1-e^{-\beta\omega}} e^{-\omega(it+1/\omega_c)} d\omega \tag{S72}$$

$$= \frac{1}{\beta^{d+1}} \int_0^\infty \frac{q^d e^{-qz}}{1-e^{-q}} dq \qquad [q = \beta\omega \text{ and } z = \frac{it}{\beta} + \frac{1}{\beta\omega_c}] \tag{S73}$$

$$= \frac{(-1)^{d+1}}{\beta^{d+1}} \psi^{(d)}(z), \tag{S74}$$

where $\psi^{(d)}(z)$ is the polygamma function defined as $\psi^n(z) = \dfrac{d^{m+1}}{dz^{m+1}} \ln\Gamma(z)$, where $\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx$ is the Gamma function. So $\psi^n(z) = \int_0^\infty \frac{q^n e^{-qz}}{1-e^{-q}} dq$.
Following similar arguments we have that

$$\mathcal{I}_2 = \frac{(-1)^{d+1}}{\beta^{d+1}} \psi^{(d)}(1 + \frac{1}{\beta\omega_c} - \frac{it}{\beta}). \tag{S75}$$

Thus the bath correlation function is

$$F_{ii}(t) = \frac{(-1)^{d+1} \eta g^2 \omega_c^{1-d}}{\beta^{d+1}} \Big[\psi^{(d)}(\frac{1}{\beta\omega_c} + \frac{it}{\beta}) + \psi^{(d)}(1 + \frac{1}{\beta\omega_c} - \frac{it}{\beta})\Big]. \tag{S76}$$

We shall assume that the quantity $\beta\omega_c >> 1$ and expand the polygamma functions in Eq. (S76) according to Taylor

series. Firstly, observe that $\frac{d^m}{dz^m}\psi^n(z) = \psi^{m+n}(z)$. Then,

$$\psi^{(d)}\left(\frac{1}{\beta\omega_c} + \frac{it}{\beta}\right) = \psi^{(d)}\left(\frac{it}{\beta}\right) + \sum_{n=1}^{\infty} \frac{\psi^{(n+d)}\left(\frac{it}{\beta}\right)}{(\beta\omega_c)^n n!} \text{ and,} \tag{S77}$$

$$\psi^{(d)}\left(1 + \frac{1}{\beta\omega_c} - \frac{it}{\beta}\right) = \psi^{(d)}\left(1 - \frac{it}{\beta}\right) + \sum_{n=1}^{\infty} \frac{\psi^{(n+d)}\left(1 - \frac{it}{\beta}\right)}{(\beta\omega_c)^n n!}. \tag{S78}$$

For simplicity, henceforth we shall concern ourselves with the case where the bath is ohmic ($d = 1$) and make statements for d in general at the end. Thus combining Eqs. (S76), (S77) and (S78) we have that the bath correlation function is

$$F_{ii}(t) = \frac{\eta g^2}{\beta^2}\left[\psi^{(1)}\left(\frac{it}{\beta}\right) + \psi^{(1)}\left(1 - \frac{it}{\beta}\right) + \sum_{n=1}^{\infty} \frac{\psi^{(n+1)}\left(\frac{it}{\beta}\right) + \psi^{(n+1)}\left(1 - \frac{it}{\beta}\right)}{(\beta\omega_c)^n n!}\right]. \tag{S79}$$

Now we shall simplify Eq. (S79) using a couple of properties of polygamma functions. Let us state these two properties first.

$$\psi^{(n)}(1-z) + (-1)^{(n-1)}\psi^n(z) = (-1)^n \pi \frac{d^n}{dz^n}\cot(\pi z) \tag{S80}$$

$$\psi^{(n)}(z+1) = \psi^n(z) + \frac{(-1)^n n!}{z^{n+1}}. \tag{S81}$$

Using Eq. (S80) for $n = 1$ and $z = it/\beta$ we have that

$$\psi^{(1)}\left(\frac{it}{\beta}\right) + \psi^{(1)}\left(1 - \frac{it}{\beta}\right) = -\pi^2 \operatorname{csch}^2(\pi t/\beta). \tag{S82}$$

Also using Eq. (S81), we have that

$$\psi^{(n+1)}\left(1 - \frac{it}{\beta}\right) = \psi^{(n+1)}\left(\frac{-it}{\beta}\right) + \frac{(-1)^n n!}{(-it/\beta)^{n+2}}. \tag{S83}$$

Substituting the results of Eq. (S82) and Eq. (S83) into Eq. (S79) we obtain

$$F_{ii}(t) = \frac{\eta g^2}{\beta^2}\left[-\pi^2 \operatorname{csch}^2(\pi t/\beta) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)!}{(-it/\beta)^{n+2}(\beta\omega_c)^n n!} + \sum_{n=1}^{\infty} \frac{\psi^{(n+1)}\left(\frac{it}{\beta}\right) + \psi^{(n+1)}\left(-\frac{it}{\beta}\right)}{(\beta\omega_c)^n n!}\right] \tag{S84}$$

$$= \eta g^2\left[-\frac{\pi^2}{\beta^2} \operatorname{csch}^2(\pi t/\beta) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{(-it)^{n+2}(\omega_c)^n} + \sum_{n=1}^{\infty} \frac{\psi^{(n+1)}\left(\frac{it}{\beta}\right) + \psi^{(n+1)}\left(-\frac{it}{\beta}\right)}{(\beta\omega_c)^n n!}\right]. \tag{S85}$$

From Eq. (S85), we find that the bath correlation time depends on both β and ω_c. Assume that $\omega_c > 1$ and that we are interested in time-scales that are larger than the thermal time-scale (i.e. $t >> \beta$) implying that $\operatorname{csch}^2(\pi t/\beta) \approx e^{-2\pi t/\beta} + \mathcal{O}(e^{-4\pi t/\beta})$. In this regime the bath correlation time-scales are $\delta t \sim \mathcal{O}(\beta/2\pi)$.

VIII. LOWER BOUND ON THE OPTIMALITY OF ANALOG QUANTUM SEARCH IN THE PRESENCE OF AN ENVIRONMENT

We prove that the running time of the analog quantum search algorithm is lower bounded by $\mathcal{O}(\sqrt{n})$ in the presence of an environment of arbitrary dimension. Our derivation also shows that the running time of this algorithm cannot be improved any further by appending an ancillary space to the original search space. We follow an argument that is similar to Ref. [21].

We are given an oracular Hamiltonian, H_w that marks the search node and add to it a time dependent drive Hamiltonian, $H_D(t)$ that takes the system to the state that was marked by H_w. Let us assume that an ancillary space of

dimension M is appended to the search space (in this case of dimension n). In such a case, the oracle Hamiltonian is

$$H_w = |w\rangle \langle w| \otimes I_M, \tag{S86}$$

where I_M is the Identity matrix of dimension M. This implies that the oracle marks a node in the search space alone. If the basis states of the environment are $\{|j\rangle\}$ for $1 \leq j \leq M$, then

$$H_w = |w\rangle \langle w| \otimes \Big(\sum_{j=1}^{M} |j\rangle \langle j| \Big) \tag{S87}$$

$$= \sum_{j=1}^{M} |w\rangle \langle w| \otimes |j\rangle \langle j| . \tag{S88}$$

Notice that the oracle is in fact marking M elements in the Hilbert space spanned by the system and the environment of dimension nM. Also

$$\sum_{w} H_w = I_{nM} \tag{S89}$$

is the sum of $(nM)/M = n$ number of disjoint possible marked states in the total nM-dimensional Hilbert space. The driver Hamiltonian $H_D(t)$ acts on the total Hilbert space. Thus the total search Hamiltonian is given by

$$H_{search} = H_w + H_D(t). \tag{S90}$$

This formalism is enough to capture the scenarios where the system under consideration (the underlying graph) undergoes interactions with the environment. The driver Hamiltonian encompasses both the Hamiltonian of the environment and the interaction Hamiltonian. Assume that the initial state of the algorithm is in some pure state $|\psi_0\rangle \in \mathbb{C}^{nM}$. If the state $|w\rangle$ is marked, let us assume that after a time t we obtain the algorithm is in state $|\psi_w(t)\rangle$. Now if a different state was marked, say $|w'\rangle$ and the algorithm commenced from the same initial state $|\psi_0\rangle$, then in order to ensure sufficient distinguishability between $|w\rangle$ and $|w'\rangle$, the states $|\psi_w(t)\rangle$ and $|\psi_{w'}(t)\rangle$ should be sufficiently distinguishable. In fact for this to happen $|\psi_w(t)\rangle$ should be sufficiently different from any $|w\rangle$-independent state $|\psi(t)\rangle$. Since $H_D(t)$ is a $|w\rangle$-independent Hamiltonian and $|\psi_0\rangle$ is a $|w\rangle$-independent state, we can use $H_D(t)$ to drive $|\psi_0\rangle$ to $|\psi(t)\rangle$. In fact we want to ensure that after some large enough time T,

$$|| \, |\psi_w(T)\rangle - |\psi(T)\rangle \, ||^2 \geq \epsilon. \tag{S91}$$

Thus,

$$\sum_{w} || \, |\psi_w(T)\rangle - |\psi(T)\rangle \, ||^2 \geq n\epsilon. \tag{S92}$$

Now we intend to obtain an upper bound for the rate of change in the norm squared of the separation between the aforementioned states, i.e.

$$\frac{d}{dt} || \, |\psi_w(t)\rangle - |\psi(t)\rangle \, ||^2 = -2 \, \mathrm{Re} \frac{d}{dt} \langle \psi_w(t) | \psi(t) \rangle \tag{S93}$$

$$= 2 \, \mathrm{Im} \, \langle \psi_w(t) | H_w | \psi(t) \rangle \tag{S94}$$

$$\leq 2 || H_w \, |\psi(t)\rangle \, ||. \tag{S95}$$

Thus

$$\frac{d}{dt} \sum_{w} || \, |\psi_w(t)\rangle - |\psi(t)\rangle \, ||^2 \leq \sum_{w} || H_w \, |\psi(t)\rangle \, ||. \tag{S96}$$

Now let

$$|\psi(t)\rangle = \sum_{i=1}^{n} \sum_{j=1}^{M} a_{ij} \, |i\rangle \, |j\rangle \, , \tag{S97}$$

where $\sum_{i=1}^{n}\sum_{j=1}^{M}|a_{ij}|^2 = 1$. Thus

$$H_w\,|\psi(t)\rangle = \sum_{j=1}^{M} a_{wj}\,|w\rangle\,|j\rangle\,, \tag{S98}$$

where $a_{jw} = \langle w|i\rangle$ and $\sum_{j=1}^{M}|a_{jw}|^2 \leq 1$. Thus we have that

$$\frac{d}{dt}\sum_{w}||\,|\psi_w(t)\rangle - |\psi(t)\rangle\,||^2 \leq \sum_{w}||H_w\,|\psi(t)\rangle\,|| \leq \sqrt{n}. \tag{S99}$$

This gives the following upper bound:

$$\sum_{w}||\,|\psi_w(T)\rangle - |\psi(T)\rangle\,||^2 \leq 2\sqrt{n}T. \tag{S100}$$

Combining Eq. (S92) and Eq. (S100), we obtain that

$$T \geq \frac{\sqrt{n}\epsilon}{2}. \tag{S101}$$

[1] David P. DiVincenzo. The physical implementation of quantum computation. *Fortschritte der Physik*, 48, 2000.

[2] G. Massimo Palma, Kalle-Antti Suominen, and Artur K. Ekert. Quantum computers and dissipation. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 452, 1996.

[3] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010.

[4] Austin G. Fowler, Matteo Mariantoni, John M. Martinis, and Andrew N. Cleland. Surface codes: Towards practical large-scale quantum computation. *Phys. Rev. A*, 86:032324, Sep 2012.

[5] Edward Farhi, Jeffrey Goldstone, Sam Gutmann, Joshua Lapan, Andrew Lundgren, and Daniel Preda. A quantum adiabatic evolution algorithm applied to random instances of an np-complete problem. *Science*, 292, 2001.

[6] Dorit Aharonov, Wim Van Dam, Julia Kempe, Zeph Landau, Seth Lloyd, and Oded Regev. Adiabatic quantum computation is equivalent to standard quantum computation. *SIAM review*, 50, 2008.

[7] Andrew M. Childs. Universal computation by quantum walk. *Physical Review Letters*, 102:180501, 2009.

[8] Andrew M Childs, David Gosset, and Zak Webb. Universal computation by multiparticle quantum walk. *Science*, 339, 2013.

[9] Stephen P Jordan, Edward Farhi, and Peter W Shor. Error-correcting codes for adiabatic quantum computation. *Physical Review A*, 74(5):052322, 2006.

[10] Kevin C Young, Mohan Sarovar, and Robin Blume-Kohout. Error suppression and error correction in adiabatic quantum computation: Techniques and challenges. *Physical Review X*, 3(4):041013, 2013.

[11] Mohan Sarovar and Kevin C Young. Error suppression and error correction in adiabatic quantum computation: non-equilibrium dynamics. *New Journal of Physics*, 15(12):125032, 2013.

[12] Masoud Mohseni, Patrick Rebentrost, Seth Lloyd, and Alan Aspuru-Guzik. Environment-assisted quantum walks in photosynthetic energy transfer. *The Journal of chemical physics*, 129(17):174106, 2008.

[13] Martin B Plenio and Susana F Huelga. Dephasing-assisted transport: quantum networks and biomolecules. *New Journal of Physics*, 10(11):113019, 2008.

[14] Patrick Rebentrost, Masoud Mohseni, Ivan Kassal, Seth Lloyd, and Alán Aspuru-Guzik. Environment-assisted quantum transport. *New Journal of Physics*, 11(3):033003, 2009.

[15] Masoud Mohseni, Yasser Omar, Gregory S Engel, and Martin B Plenio. *Quantum effects in biology*. Cambridge University Press, 2014.

[16] Filippo Caruso, Alex W Chin, Animesh Datta, Susana F Huelga, and Martin B Plenio. Highly efficient energy excitation transfer in light-harvesting complexes: The fundamental role of noise-assisted transport. *The Journal of Chemical Physics*, 131(10):105106, 2009.

[17] Jianlan Wu, Robert J Silbey, and Jianshu Cao. Generic mechanism of optimal energy transfer efficiency: A scaling theory of the mean first-passage time in exciton systems. *Physical review letters*, 110(20):200402, 2013.

[18] Leonardo Novo, Masoud Mohseni, and Yasser Omar. Disorder-assisted quantum transport in suboptimal decoherence regimes. *Scientific reports*, 6:18142, 2016.

[19] Philip W Anderson. Absence of diffusion in certain random lattices. *Physical review*, 109(5):1492, 1958.

[20] Marco del Rey, Alex W Chin, Susana F Huelga, and Martin B Plenio. Exploiting structured environments for efficient energy transfer: the phonon antenna mechanism. *The journal of physical chemistry letters*, 4(6):903–907, 2013.

[21] Edward Farhi and Sam Gutmann. Analog analogue of a digital quantum computation. *Physical Review A*, 57:2403, 1998.

[22] Andrew M. Childs and Jeffrey Goldstone. Spatial search by quantum walk. *Physical Review A*, 70:022314, 2004.

[23] Anthony J Leggett, S Chakravarty, AT Dorsey, Matthew PA Fisher, Anupam Garg, and W Zwerger. Dynamics of the dissipative two-state system. *Reviews of Modern Physics*, 59(1):1, 1987.

[24] Crispin Gardiner and Peter Zoller. *Quantum noise: a handbook of Markovian and non-Markovian quantum stochastic methods with applications to quantum optics*, volume 56. Springer Science & Business Media, 2004.

[25] Heinz-Peter Breuer and Francesco Petruccione. *The theory of open quantum systems*. Oxford University Press on Demand, 2002.

[26] Frank Verstraete, Michael M Wolf, and J Ignacio Cirac. Quantum computation and quantum-state engineering driven by dissipation. *Nature Physics*, 5(9):633–636, 2009.

[27] Andrew M Childs, Edward Farhi, and John Preskill. Robustness of adiabatic quantum computation. *Physical Review A*, 65(1):012322, 2001.

[28] Mohammad Amin, Peter J Love, and Colin JS Truncik. Thermally assisted adiabatic quantum computation. *Physical review letters*, 100(6):060503, 2008.

[29] Ines De Vega, Mari Carmen Banuls, and A Pérez. Effects of dissipation on an adiabatic quantum search algorithm. *New Journal of Physics*, 12(12):123010, 2010.

[30] Tameem Albash, Sergio Boixo, Daniel A Lidar, and Paolo Zanardi. Quantum adiabatic markovian master equations. *New Journal of Physics*, 14(12):123016, 2012.

[31] Neil G Dickson, MW Johnson, MH Amin, R Harris, F Altomare, AJ Berkley, P Bunyk, J Cai, EM Chapple, P Chavez, et al. Thermally assisted quantum annealing of a 16-qubit problem. *Nature communications*, 4:1903, 2013.

[32] Dominik S Wild, Sarang Gopalakrishnan, Michael Knap, Norman Y Yao, and Mikhail D Lukin. Adiabatic quantum search in open systems. *Physical review letters*, 117(15):150501, 2016.

[33] This energy at the marked node implies that the quantum simulation of $|w\rangle\langle w|$ for time t would correspond to $\mathcal{O}(t)$ queries to the standard Grover oracle.

[34] Neil Shenvi, Kenneth R Brown, and K Birgitta Whaley. Effects of a random noisy oracle on search algorithm complexity. *Physical Review A*, 68(5):052313, 2003.

[35] Karl Blum. *Density matrix theory and applications*, volume 64. Springer Science & Business Media, 2012.

[36] Theodore J Yoder, Guang Hao Low, and Isaac L Chuang. Fixed-point quantum search with an optimal number of queries. *Physical review letters*, 113(21):210501, 2014.

[37] Alexander M Dalzell, Theodore J Yoder, and Isaac L Chuang. Fixed-point adiabatic quantum search. *Physical Review A*, 95(1):012311, 2017.

[38] Leonardo Novo, Shantanav Chakraborty, Masoud Mohseni, Hartmut Neven, and Yasser Omar. Systematic dimensionality reduction for quantum walks: Optimal spatial search and transport on non-regular graphs. *Scientific Reports*, 5:13304, 2015.

[39] Shantanav Chakraborty, Leonardo Novo, Andris Ambainis, and Yasser Omar. Spatial search by quantum walk is optimal for almost all graphs. *Phys. Rev. Lett.*, 116:100501, Mar 2016.

[40] Shantanav Chakraborty, Leonardo Novo, Serena Di Giorgio, and Yasser Omar. Optimal quantum spatial search on random temporal networks. *arXiv preprint arXiv:1701.04392*, 2017.

[41] Inés de Vega and Daniel Alonso. Dynamics of non-markovian open quantum systems. *Reviews of Modern Physics*, 89(1):015001, 2017.