# Dynamical phase transitions at finite temperature from fidelity and interferometric Loschmidt echo induced metrics

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We study finite-temperature Dynamical Quantum Phase Transitions (DQPTs) by means of the fidelity and the interferometric Loschmidt Echo (LE) induced metrics. We analyse the associated dynamical susceptibilities (Riemannian metrics), and derive analytic expressions for the case of two-band Hamiltonians. At zero temperature the two quantities are identical, nevertheless, at finite temperatures they behave very differently. Using the fidelity LE, the zero temperature DQPTs are gradually washed away with temperature, while the interferometric counterpart exhibits finite-temperature PTs. We analyse the physical differences between the two finite-temperature LE generalisations, and argue that, while the interferometric one is more sensitive and can therefore provide more information when applied to genuine quantum (microscopic) systems, when analysing many-body macroscopic systems, the fidelity-based counterpart is more suitable quantity to study. Finally, we apply the previous results to two representative models of topological insulators in 1D and 2D.

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## I. INTRODUCTION

Equilibrium phase transitions (PTs) are characterised by non-analyticities in some thermodynamic quantities that describe a system as its temperature is varied. Quantum phase transitions (QPTs) [1], traditionally described by Landau theory [2], occur when we adiabatically change a physical parameter of the system at zero temperature, i.e., the transition is driven by purely quantum fluctuations. A particularly interesting case arises when one studies QPTs in the context of topological phases of matter [3–7]. They are quite different from the standard QPTs, since they do not involve symmetry breaking and are characterised by global order parameters [8]. These novel phases of matter, which include topological insulators and superconductors [9–11], have potentially many applications in emerging fields such as spintronics, photonics or quantum computing. Although many of their remarkable properties have traditionally been studied at zero temperature, there has been a great effort to generalise these phases from pure to mixed states, and finite temperatures [12–23].

The real time evolution of closed quantum systems out of equilibrium has some surprising similarities with thermal phase transitions, as noticed by Heyl, Polkovnikov and Kehrein [24]. They coined the term DQPTs to describe the non-analytic behaviour of certain dynamical observables after a sudden quench in one of the parameters of the Hamiltonian. Since then, the study of DQPTs became an active field of research and a lot of progress has been achieved in comparing and connecting them to the equilibrium PTs [25, 26]. Along those lines, there exist several studies of DQPTs for systems featuring non-trivial topological properties [27–30]. The figure of merit in the study of DQPTs is the LE and its derivatives, which have been extensively used in the analysis of quantum criticality [31–35] and quantum quenches [36]. At finite temperature, generalisations of the zero temperature LE were proposed, based on the mixed-state Uhlmann fidelity [35, 37], and the interferometric mixed-state geometric phase [38, 39]. Fidelity is a measure of state distinguishability, which has been employed numerous times in the study of PTs [32, 40–43], while the interferometric mixed-state geometric phase was introduced in [44].

In this paper, we analyse the finite temperature behaviour of quenched systems in terms of both the *fidelity* and the *interferometric LEs*. In particular, we discuss the existence of finite temperature DPTs in two-band Hamiltonians for the 1D Su–Schrieffer–Heeger (SSH) model and the 2D Massive Dirac Hamiltonian (MD), known to exhibit zero temperature topological QPTs. The two approaches give opposite predictions: the fidelity LE shows a gradual disappearance of DPTs as the temperature increases, while the interferometric LE indicates their persistence at finite temperature (consistent with recent studies on interferometric LE [38, 39]). We analyse the reasons for such different behaviours, in terms of the metrics (susceptibilities) induced by the fidelity and the interferometric LEs. The former quantifies state distinguishability in terms of measurements of physical properties, while the latter quantifies the effects of quantum channels acting upon a state. In addition, interferometric experiments that are suitable for genuine (microscopic) quantum systems involve coherent superpositions of two states, which could be, in the case of many-body macroscopic systems, experimentally infeasible with the current technology.

The paper is organised as follows: In Section II, after introducing some preliminaries about the general features of DQPTs, we perform a detailed analysis of the two LE generalisations, and compare them with respect to the study of DQPTs. To confirm this analysis, in Section III we present quantitative results for the fidelity-induced first time derivative of the rate function in the case of the SSH topological insulator and MD model, which captures the physics of a 2D Chern insulator [45]. Section IV is devoted to conclusions.

## II. DYNAMICAL (QUANTUM) PHASE TRANSITIONS AND ASSOCIATED SUSCEPTIBILITIES

As mentioned in Introduction, the authors in [24] introduce the concept of DQPTs and illustrate their properties on the case of the transverse-field Ising model. They observe a similarity between the partition function of a quantum system in equilibrium, $Z(\beta) = \mathrm{Tr}(e^{-\beta H})$, and the overlap amplitude of some time-evolved initial quantum state $|\psi_i\rangle$ with itself, $G(t) = \langle\psi_i|e^{-iHt}|\psi_i\rangle$. During a temperature-driven PT, the abrupt change of the properties of a system are indicated by the non-analyticity of the free energy density $f(\beta) = -\lim_{N\to\infty}\frac{1}{N}\ln Z(\beta)$ at the critical temperature ($N$ being the number of degrees of freedom). It is then possible to establish an analogy with the case of the real time evolution of a quantum system out of equilibrium, by considering the rate function

$$g(t) = -\frac{1}{N}\log|G(t)|^2, \tag{1}$$

where $|G(t)|^2$ is a mixed-state LE, as we detail below. The rate function $g(t)$ may exhibit non-analyticities at some critical times $t_c$, after a quantum quench. This phenomenon is termed DPT.

We study DPTs for mixed states using the fidelity and the interferometric LEs. We first investigate the relation between the two approaches for DQPTs at zero temperature. More concretely, we perform analytical derivations of the corresponding susceptibilities in the general case of a family of static Hamiltonians, parametrised by some smooth manifold $M$, $\{H(\lambda) : \lambda \in M\}$.

### DQPTs for pure states

At zero temperature, the LE $G(t)$ from (1) between the ground state for $\lambda = \lambda_i \in M$ and the evolved state with respect to the Hamiltonian for $\lambda = \lambda_f \in M$ is given by the fidelity between the two states

$$\mathcal{F}(t;\lambda_f,\lambda_i) \equiv |\langle\psi(\lambda_i)|e^{-itH(\lambda_f)}|\psi(\lambda_i)\rangle|. \tag{2}$$

For $\lambda_i = \lambda_f$, the fidelity is trivial, since the system remains in the same state. Fixing $\lambda_i \equiv \lambda$ and $\lambda_f = \lambda + \delta\lambda$, with $\delta\lambda << 1$, in the $t \to \infty$ limit Eq. (2) is nothing but the familiar $S$-matrix with an unperturbed Hamiltonian $H(\lambda)$ and an interaction Hamiltonian $V(\lambda)$, which is approximated by

$$V(\lambda) \equiv H(\lambda_f) - H(\lambda) \approx \frac{\partial H}{\partial\lambda^a}(\lambda)\delta\lambda^a. \tag{3}$$

After applying standard perturbation theory techniques (see Appendix), we obtain

$$\mathcal{F}(t;\lambda_f,\lambda) \approx 1 - \chi_{ab}(t;\lambda)\delta\lambda^a\delta\lambda^b, \tag{4}$$

where the dynamical susceptibility $\chi_{ab}(t;\lambda)$ is given by

$$\chi_{ab}(t;\lambda) = \int_0^t\int_0^t dt_2 dt_1 \left(\frac{1}{2}\langle\{V_a(t_2),V_b(t_1)\}\rangle - \langle V_a(t_2)\rangle\langle V_b(t_1)\rangle\right), \tag{5}$$

with $V_a(t,\lambda) = e^{itH(\lambda)}\partial H/\partial\lambda^a(\lambda)e^{-itH(\lambda)}$ and $\langle * \rangle = \langle\psi(\lambda)| * |\psi(\lambda)\rangle$. The family of symmetric tensors $\{ds^2(t) = \chi_{ab}(t,\lambda)d\lambda^a d\lambda^b\}_{t\in\mathbb{R}}$ defines a family of metrics in the manifold $M$, which can be seen as pullback metrics of the Bures metric (Fubini-Study metric) in the manifold of pure states [46]. Specifically, at time $t$, the pullback is given by the map $\Phi_t : \lambda_f \mapsto e^{-itH(\lambda_f)}|\psi(\lambda)\rangle\langle\psi(\lambda)|e^{itH(\lambda_f)}$, evaluated at $\lambda_f = \lambda$.

### Generalisations at finite temperatures

The generalisation of DQPTs to mixed states is not unique. There are several ways to construct a LE for a general density matrix. In what follows, we derive two finite temperature generalisations, such that they have the same zero temperature limit $T = 1/\beta \to 0$.

#### *A) Fidelity Loschmidt Echo*

First, we introduce the *fidelity LE* between the state $\rho(\beta;\lambda_i) = e^{-\beta H(\lambda_i)}/\mathrm{Tr}\{e^{-\beta H(\lambda_i)}\}$ and the one evolved by the unitary operator $e^{-itH(\lambda_f)}$ as

$$\mathcal{F}(t,\beta;\lambda_i,\lambda_f) = F(\rho(\beta;\lambda_i), e^{-itH(\lambda_f)}\rho(\beta;\lambda_i)e^{itH(\lambda_f)}), \tag{6}$$

where $F(\rho,\sigma) = \mathrm{Tr}\{\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\}$ is the quantum fidelity between arbitrary mixed states $\rho$ and $\sigma$. For $\lambda_f$ close to $\lambda_i = \lambda$, we can write

$$\mathcal{F}(t,\beta;\lambda_f,\lambda) \approx 1 - \chi_{ab}(t,\beta;\lambda)\delta\lambda^a\delta\lambda^b, \tag{7}$$

with $\chi_{ab}(t,\beta,\lambda)$ being the *Dynamical Fidelity Susceptibility* (DFS). Notice that $\lim_{\beta\to\infty}\chi_{ab}(t,\beta;\lambda) = \chi_{ab}(t;\lambda)$, where $\chi_{ab}(t;\lambda)$ is given by Eq. (5). At time $t$ and inverse temperature $\beta$, we have a map $\Phi_{(t,\beta)} : \lambda_f \mapsto e^{-itH(\lambda_f)}\rho(\beta;\lambda)e^{itH(\lambda_f)}$. The 2-parameter family of metrics defined by $ds^2(\beta,t) = \chi_{ab}(t,\beta;\lambda)d\lambda^a d\lambda^b$ is the pullback by $\Phi_{(t,\beta)}$ of the Bures metric on the manifold of full-rank density operators, evaluated at $\lambda_f = \lambda$.

The fidelity LE is closely related to the Uhlmann connection: $F(\rho_1,\rho_2)$ equals the overlap between purifications $W_1$ and $W_2$, $\langle W_1, W_2\rangle = \mathrm{Tr}\left\{W_1^\dagger W_2\right\}$, satisfying discrete parallel transport condition (see, for instance, [47]).

### *B) Interferometric Loschmidt Echo*

Here, we consider an alternative definition of the LE for mixed states [$G(t)$ from Eq. (1)]. In particular, we define the *interferometric LE* as

$$\mathcal{L}(t,\beta;\lambda_f,\lambda_i) = \left|\frac{\mathrm{Tr}\left\{e^{-\beta H(\lambda_i)}e^{itH(\lambda_i)}e^{-itH(\lambda_f)}\right\}}{\mathrm{Tr}\left\{e^{-\beta H(\lambda_i)}\right\}}\right|. \quad (8)$$

The $e^{itH(\lambda_i)}$ factor does not appear at zero temperature, since it just gives a phase which is canceled by taking the absolute value. This differs from previous treatments in the literature [29] (see Section 5.5.4 of [46], where the variation of the interferometric phase, $\mathrm{Tr}\{\rho_0 e^{-itH}\}$, exposes this structure). However, it is convenient to introduce it in order to have the usual form of the perturbation expansion, as will become clear later.

For $\lambda_f$ close to $\lambda_i = \lambda$, we get

$$\mathcal{L}(t,\beta;\lambda_f,\lambda) \approx \left|\frac{\mathrm{Tr}\left\{e^{-\beta H(\lambda)}\mathcal{T}e^{-i\int_0^t dt' V_a(t,\lambda)\delta\lambda^a}\right\}}{\mathrm{Tr}\left\{e^{-\beta H(\lambda)}\right\}}\right|, \quad (9)$$

so that the perturbation expansion goes as in Eq. (5), yielding

$$\mathcal{L}(t,\beta;\lambda_f,\lambda) \approx 1 - \tilde{\chi}_{ab}(t,\beta;\lambda)\delta\lambda^a\delta\lambda^b, \quad (10)$$

with the dynamical susceptibility given by

$$\tilde{\chi}_{ab}(t,\beta;\lambda) = \int_0^t\int_0^t dt_2 dt_1 \left(\frac{1}{2}\langle\{V_a(t_2),V_b(t_1)\}\rangle - \langle V_a(t_2)\rangle\langle V_b(t_1)\rangle\right), \quad (11)$$

where $\langle * \rangle = \mathrm{Tr}\{e^{-\beta H(\lambda)}*\}/\mathrm{Tr}\{e^{-\beta H(\lambda)}\}$. Notice that Eqs (11) and (5) are formally the same with the average over the ground state replaced by the thermal average. This justifies the extra $e^{itH(\lambda_i)}$ factor. Since this susceptibility comes from the interferometric LE, we call it *Dynamical Interferometric Susceptibility* (DIS). The quantity $ds^2(\beta,t) = \tilde{\chi}_{ab}(t,\beta;\lambda)d\lambda^a d\lambda^b$ defines a 2-parameter family of metrics over the manifold $M$, except that they cannot be seen as pullbacks of metrics on the manifold of density operators with full rank. However, it can be interpreted as the pullback by a map from $M$ to the unitary group associated with the Hilbert space of a particular Riemannian metric. For a detailed analysis, see the last subsection of the Appendix. Additionally, we point out that this version of LE is related to the interferometric geometric phase introduced by Sjöqvist *et. al* [44, 48].

### Two-band systems

Many representative examples of topological insulators and superconductors can be described by effective two-band Hamiltonians. Therefore, we derive close expressions of the previously introduced dynamical susceptibilities for topological systems within this class. For simplicity, we consider two-level systems. Generalization to many-body two-band systems is then straightforward.

The general form of such Hamiltonians is $\{H(\lambda) = \vec{x}(\lambda)\cdot\vec{\sigma} : \lambda \in M\}$, where $\vec{\sigma}$ is the Pauli vector. The interaction Hamiltonian $V(\lambda)$, introduced in Eq. (2), casts the form

$$V(\lambda) \approx \left(\frac{\partial\vec{x}}{\partial\lambda^a}\cdot\vec{\sigma}\right)\delta\lambda^a.$$

It is convenient to decompose $\partial\vec{x}/\partial\lambda^a$ into one component perpendicular to $\vec{x}$ and one parallel to it:

$$\frac{\partial\vec{x}}{\partial\lambda^a} = \left(\frac{\partial\vec{x}}{\partial\lambda^a}\right)^{\perp} + \left(\frac{\partial\vec{x}}{\partial\lambda^a}\right)^{\parallel} = \vec{t}_a + \vec{n}_a.$$

The first term is tangent, in $\mathbb{R}^3$, at $\vec{x}(\lambda)$, to a sphere of constant radius $r = |\vec{x}(\lambda)|$. Hence, this kind of perturbations do not change the spectrum of $H$, only its eigenbasis. The second term is a variation of the length of $\vec{x}$ and hence, it changes the spectrum of $H$, while keeping the eigenbasis fixed. The DFS and the DIS are given by (for details of the derivation, see Appendix)

$$\chi_{ab} = \tanh^2(\beta|\vec{x}(\lambda)|)\frac{\sin^2(|\vec{x}(\lambda)|t)}{|\vec{x}(\lambda)|^2}\vec{t}_a\cdot\vec{t}_b \quad (12)$$

$$\tilde{\chi}_{ab} = \frac{\sin^2(|\vec{x}(\lambda)|t)}{|\vec{x}(\lambda)|^2}\vec{t}_a\cdot\vec{t}_b + t^2(1-\tanh^2(\beta|\vec{x}(\lambda)|))\vec{n}_a\vec{n}_b. \quad (13)$$

While the DIS (13) depends on the variation of both the spectrum and the eigenbasis of the Hamiltonian, the DFS (12) depends only on the variations which preserve the spectrum, i.e., changes in the eigenbasis. This is very remarkable, in general the fidelity between two quantum states, being their distinguishability measure, does depend on both the variations of the spectrum and the eigenbasis. In our particular case of a quenched system, the eigenvalues are preserved (see Eq. (6)), as the system is subject to a unitary evolution. The tangential components of both susceptibilities are modulated by the function $\sin^2(Et)/E^2$, where $E$ is the gap. This captures the *Fisher zeros*, i.e., the zeroes of the (dynamical) partition function which here is given by the Fidelity $\mathcal{F}$ from (6), see [49–51]. Observe that whenever $t = (2n+1)\pi/2E$,

$n \in \mathbb{Z}$, this factor is maximal and hence, both LEs decrease abruptly. The difference between the two susceptibilities is given by

$$\tilde{\chi}_{ab} - \chi_{ab}$$
$$= (1 - \tanh^2(\beta|\vec{x}(\lambda)|)) \left( \frac{\sin^2(|\vec{x}(\lambda)|t)}{|\vec{x}(\lambda)|^2} \vec{t}_a \cdot \vec{t}_b + t^2 \vec{n}_a \cdot \vec{n}_b \right).$$

The quantity $(1 - \tanh^2(\beta E))$ is nothing but the static susceptibility, see [52]. Therefore, the difference between DIS and DFS is modulated by the static susceptibility at finite temperature. For small times, $\sin^2(Et)/E^2 \approx t^2$, and we have

$$\tilde{\chi}_{ab} - \chi_{ab} \approx (1 - \tanh^2(\beta|\vec{x}(\lambda)|) t^2 \frac{\partial \vec{x}}{\partial \lambda^a} \cdot \frac{\partial \vec{x}}{\partial \lambda^b},$$

while for larger times, $\sin^2(Et)/E^2 \approx \pi t \delta(E)$,

$$\tilde{\chi}_{ab} - \chi_{ab}$$
$$\approx \pi t \delta(|\vec{x}(\lambda)|) \vec{t}_a \cdot \vec{t}_b + t^2 (1 - \tanh^2(\beta|\vec{x}(\lambda)|)) \vec{n}_a \cdot \vec{n}_b.$$

To illustrate the relationship between the two susceptibilities, in FIG 1 we plotted the modulating function for the tangential components of both. We observe that at zero temperature they coincide. As the temperature increases, in the case of the fidelity LE, the gap vanishing points become less important. On the contrary, for the interferometric LE, the associated tangential part of the susceptibility does not depend on temperature, thus the gap vanishing points remain prominent. The DFS from Eq. (12) thus predicts gradual smearing of critical behaviour, consistent with previous findings that showed the absence of phase transitions at finite temperatures in the static case [14, 15]. The DIS from Eq.(13) has a tangential term that is not coupled to the temperature, persisting at higher temperatures and giving rise to abrupt changes in the finite temperature system's behaviour. This is also consistent with previous studies in the literature, where DPTs were found even at finite temperatures [38, 39]. Additionally, the interferometric LE depends on the normal components of the variation of $\vec{x}$.

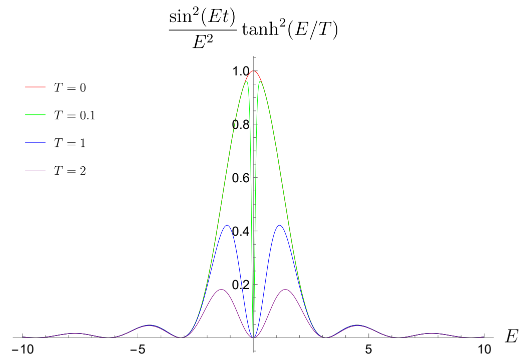


FIG. 1. The susceptibility modulating function for the tangential components at $t = 1$.

### Comparing the two approaches

The above analysis of the two dynamical susceptibilities (metrics) reflects the essential difference between the two distinguishability measures, one based on the fidelity, the other on interferometric experiments. From the quantum information theoretical point of view, the two quantities can be interpreted as distances between *states*, or between *processes*, respectively. The Hamiltonian evaluated at a certain point of parameter space $M$ defines the macroscopic phase. Associated to it we have thermal states and unitary processes. The fidelity LE is obtained from the Bures distance between a thermal state $\rho_1$ in phase 1 and the one obtained by unitarily evolving this state, $U_2 \rho U_2^\dagger$, with $U_2$ associated to phase 2. Given a thermal state $\rho_1$ prepared in phase 1, the interferometric LE is obtained from the distance between two unitary processes $U_1$ and $U_2$ (defined modulo a phase factor), associated to phases 1 and 2.

The quantum fidelity between two states $\rho$ and $\sigma$ is in fact the classical fidelity between the probability distributions $\{p_i\}$ and $\{q_i\}$, and is obtained by performing an optimal measurement $\mathscr{M}$ on the two states $\rho$ and $\sigma$, respectively. For that reason, one can argue that the fidelity is capturing all order parameters (i.e., measurements) through its optimal observables $\mathscr{M}$. On the other hand, the interferometric phase is based on some interferometric experiment to distinguish two states, $\rho$ and $U\rho U^\dagger$: it measures how the intensities at the outputs of the interferometer are affected by applying $U$ to only one of its arms [44]. Therefore, to set up such an experiment, one does not need to know the state $\rho$ that enters the interferometer, as only the knowledge of $U$ is required. This is a different type of experiment, not based on the observation of any physical property of a system. It is analogous to comparing two masses with weighing scales, which would show the same difference of $\Delta m = m_1 - m_2$, regardless of how large the two masses $m_1$ and $m_2$ are. For that reason, interferometric distinguishability is more sensitive than the fidelity (fidelity depends on more information, not only how much the two states are different, but in which aspects this difference is observable). In terms of experimental feasibility, the fidelity is more suitable for the study of many-body macroscopic systems and phenomena, while the interferometric measurements provide a more detailed information on genuinely quantum (microscopic) systems. Finally, interferometric experiments involve coherent superpositions of two states. Therefore, when applied to many-body systems, one would need to create genuine Schrödinger cat-like states, which goes beyond the current, and any foreseeable, technology (and could possibly be forbidden by more fundamental laws of physics, see for example objective collapse theories [53]).

## III. DPTS OF TOPOLOGICAL INSULATORS AT FINITE TEMPERATURES

In order to confirm the main result of our paper from Section II, obtained for generic two-band systems, we study the fidelity LE on concrete examples of two topological insulators (as noted before, the analogous study for the interferometric LE on concrete examples has already been performed, and is consistent with our findings [38, 54]). In particular, we present quantitative results obtained for the first derivative of the rate function, $dg/dt$, where $g(t) = -\frac{1}{N}\log\mathcal{F}$, and $\mathcal{F}$ is given by Eq. (6). The quantity $dg/dt$ is the figure of merit in the study of the DQPTs, therefore we present the respective results that confirm the previous study: the generalisation of the LE with respect to the fidelity shows the absence of finite − temperature dynamical PTs. We consider two paradigmatic models of topological insulators, namely the SSH [55] and the MD [45] models.

### SSH model (1D)

The SSH model was introduced in [55] to describe polyacetilene, and it was later found to describe diatomic polymers [56]. In momentum space, the Hamiltonian for this model is of the form $H(k,m) = \vec{x}(k,m)\cdot\vec{\sigma}$, with $m$ being the parameter that drives the static PT. The vector $\vec{x}(k,m)$ is given by:

$$\vec{x}(k,m) = (m + \cos(k), \sin(k), 0).$$

By varying $m$ we find two distinct topological regimes. For $m < m_c = 1$ the system is in a non-trivial phase with winding number 1, while for $m > m_c = 1$ the system is in a topologically trivial phase with winding number 0.

We consider both cases in which we go from a trivial to a topological phase and vice versa (FIGs 2 and 3, respectively). We notice that non-analyticities of the first derivative appear at zero temperature, and they are smeared out for higher temperatures.

### MD model (2D)

The Massive Dirac model (MDM) captures the physics of a 2D Chern insulator [45], and shows different topologically distinct phases. In momentum space, the Hamiltonian for the MDM is of the form $H(\vec{k},m) = \vec{x}(\vec{k},m)\cdot\vec{\sigma}$, with $m$ being the parameter that drives the static PT. The vector $\vec{x}(\vec{k},m)$ is given by

$$\vec{x}(\vec{k},m) = (\sin(k_x), \sin(k_y), m - \cos(k_x) - \cos(k_y)).$$

By varying $m$ we find four different topological regimes:

- For $-\infty < m < m_{c_1} = -2$ it is trivial (the Chern number is zero) – Regime I

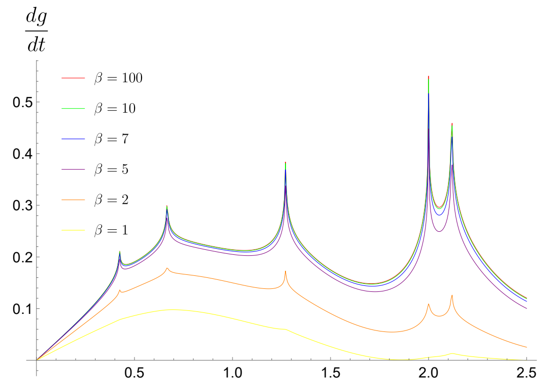


FIG. 2. We plot the time derivative of the rate function, $dg/dt$, as a function of time for different values of the inverse temperature $\beta = 1/T$. We consider a quantum quench from a trivial phase ($m = 1.2$) to a topological phase ($m = 0.8$).

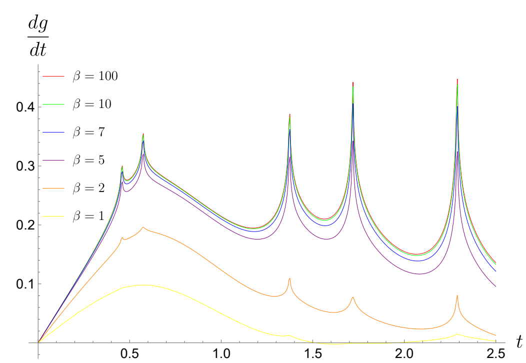


FIG. 3. The time derivative of the rate function, $dg/dt$, as a function of time for different values of the inverse temperature. The quench is from a topological ($m = 0.8$) to a trivial $phase$($m = 1.2$).

- For $-2 = m_{c_1} < m < m_{c_2} = 0$ it is topological (the Chern number is $-1$) – Regime II
- For $0 = m_{c_2} < m < m_{c_3} = 2$ it is topological (the Chern number is $+1$) – Regime III
- For $2 = m_{c_3} < m < \infty$ it is trivial (the Chern number is zero) – Regime IV

In FIGs 4,5 and 6 we plot the first derivative of the rate function $g(t)$, as a function of time for different temperatures. We only consider quenches that traverse a single phase transition point.

We observe that at zero temperature there exist non-analyticities at the critical times – the signatures of DQPTs. As we increase the temperature, these non-analyticities are gradually smeared out, resulting in smooth curves for higher finite temperatures. We note that the peak of the derivative dg(t)/dt is drifted when increasing the temperature, in analogy to the usual drift of non-dynamical quantum phase transitions at finite temperature [57].

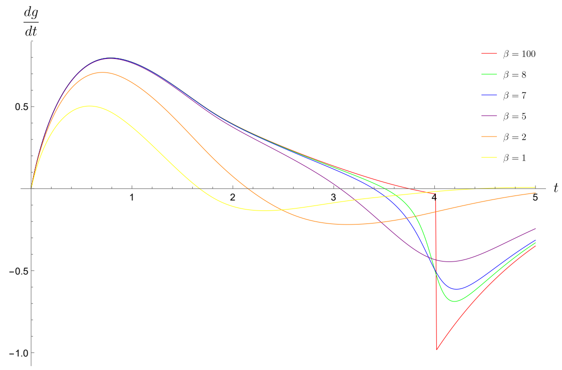


FIG. 4. The time derivative of the rate function, $dg/dt$, as a function of time for different values of the inverse temperature. We quench the system from a trivial to a topological regime (Regimes from I to II and from IV to III).

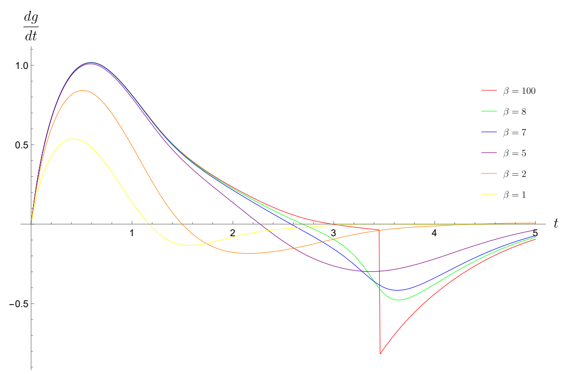


FIG. 5. The time derivative of the rate function, $dg/dt$, as a function of time for different values of the inverse temperature. The quench is from a topological to a trivial regime (Regimes from II to I and from III to IV).

Next, we proceed by considering the cases in which we cross two phase transition points, as shown in FIGs 7 and 8. At zero temperature we obtain a non-analytic behaviour, which gradually disappears for higher temperatures.

Finally, we have also studied the case in which we move inside the same topological regime from left to right and vice versa. We obtained smooth curves without non-analyticities, which we omit for the sake of briefness.

## IV. CONCLUSIONS

We analysed the fidelity and the interferometric generalisations of the LE for general mixed states, and applied them to the study of finite temperature DPTs in topological systems. At the level of dynamical susceptibilities, i.e., the metric tensors they define in the space of parameters, the two tensors emerge as pull-backs of metrics in different spaces. Using the fidelity LE, we show that the dynamical susceptibility is the pull-back of the Bures

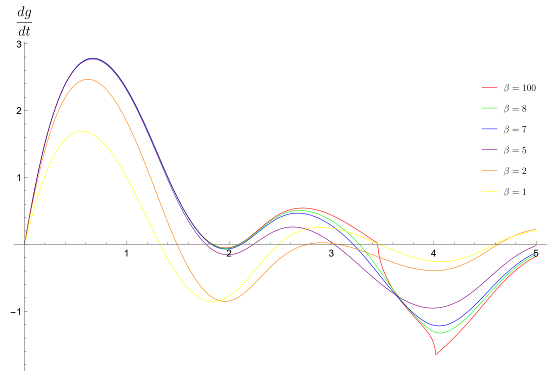


FIG. 6. The time derivative of the rate function, $dg/dt$, as a function of time for different inverse temperatures. The quantum quench is from a topological to a topological regime (Regimes from II to III and vice versa).

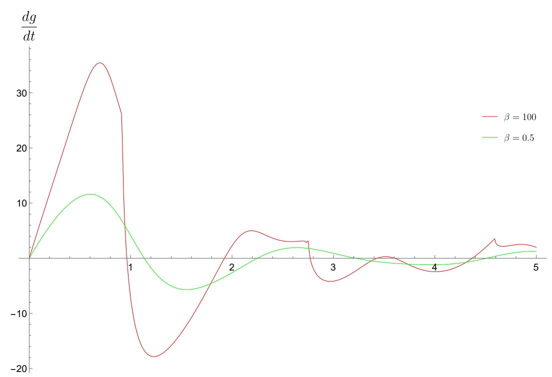


FIG. 7. The time derivative of the rate function, $dg/dt$, as a function of time for different values of the inverse temperature, in the case that we quench the system from a trivial to a topological regime (Regimes from I to III and from IV to II).

metric in the *space of density matrices*, i.e., states. On the other hand, if we make us of the interferometric LE, the dynamical susceptibility is the pull-back of a metric in the *space of unitaries* (i.e., evolutions/processes).

As discussed in Section II, the difference between the two metrics reflects the fact that the fidelity is a measure of the state distinguishability between two *given* states $\rho$ and $\sigma$ in terms of observations, while the "interferometric distinguishability" quantifies how a quantum channel (a unitary $U$) changes an *arbitrary* state $\rho$ to $U\rho U^\dagger$. Therefore, while the "interferometric distinguishability" is in general more sensitive, and thus appropriate for the study of genuine (microscopic) systems, it is the fidelity that is the most suitable for the study of many-body system phases. Moreover, interferometric experiments involve coherent superpositions of two states, which for many-body systems would require creating and manipulating genuine Schrödinger cat-like states. This goes beyond any foreseeable future technologies.

We have presented analytic expressions for the dynam-

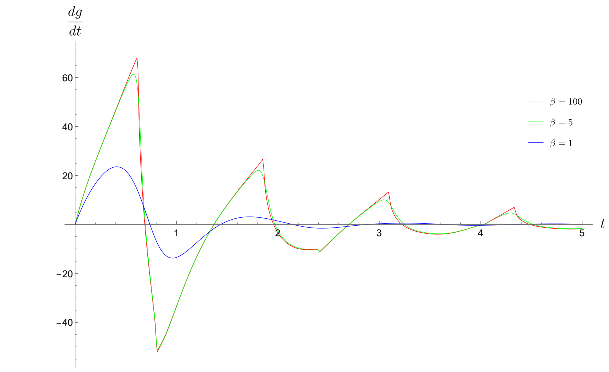


FIG. 8. The time derivative of the rate function, $dg/dt$, as a function of time for different values of the inverse temperature. The system is quenched from a topological to a trivial regime (Regimes from III to I and from II to IV).

ical susceptibilities in the case of two-band Hamiltonians. At finite temperature, the fidelity LE indicates gradual disappearance of the zero-temperature DQPTs, while the interferometric LE predicts finite-temperature DPTs. To confirm those results, we have performed finite temperature study of the first time derivative of the fidelity rate function, $dg(t)/dt$, on two representatives of topological insulators: the 1D SSH and the 2D Massive Dirac models. In perfect agreement with the general result, the fidelity-induced first derivatives gradually smear smears down with temperature, not exhibiting any critical behaviour at finite temperatures. This is consistent with recent studies of one-dimensional symmetry protected topological phases at finite temperatures [14, 15]. On the contrary, the interferometric LE exhibits critical behavior even at finite temperatures (confirming previous studies on DPTs [38, 39]).

*Note added.* During the preparation of this manuscript, we became aware of a recent related work [58].

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## APPENDIX

### Analytical Derivation of the Dynamical Susceptibilities

*Zero Temperature case*

Let $\mathscr{H}$ be a Hilbert space. Suppose we have a family of Hamiltonians $\{H(\lambda) : \lambda \in M\}$ where $M$ is a smooth compact manifold. We assume that besides a closed finite subset of $M$, $C = \{\lambda_i\}_{i=1}^n \subset M$, the Hamiltonian is gapped and the ground state subspace is one-dimensional. Locally, on $M - C$, we can find a ground state (with unit norm), described by $|\psi(\lambda)\rangle$. Take $\lambda_i \in C$, and let $U$ be an open neighbourhood containing $\lambda_i$. Of course, for sufficiently small $U$, on the open set $U - \{\lambda_i\}$ one can find a smooth assignment $\lambda \mapsto |\psi(\lambda)\rangle$. Consider a curve $[0,1] \ni s \mapsto \lambda(s) \in U$, with initial condition $\lambda(0) = \lambda_0$, such that $\lambda(s_0) = \lambda_i$ for some $s_0 \in [0,1]$. The family of Hamiltonians $H(s) := H(\lambda(s))$ is well-defined for every $s \in [0,1]$. The family of states $|\psi(s)\rangle \equiv |\psi(\lambda(s))\rangle$ is well-defined for $s \neq s_0$ and so is the ground state energy,

$$E(s) := \langle\psi(s)|H(s)|\psi(s)\rangle.$$

The overlap:

$$\mathcal{A}(s) := \langle\psi(0)|\exp(-itH(s))|\psi(0)\rangle,$$

is well-defined. We can write,

$$\exp(-itH(s)) = \exp(-itH(0))T\exp\left\{-i\int_0^t d\tau V(s,\tau)\right\}$$

If we take derivative with respect to $t$ of the equation, we find,

$$H(s) = H(0) + \exp(-itH(0))V(s,t)\exp(itH(0))$$

so,

$$V(s,t) = \exp(itH(0))(H(s) - H(0))\exp(-itH(0)).$$

We can now write, since $|\psi(0)\rangle$ is an eigenvector of $H(0)$,

$$\mathcal{A}(s) = e^{-itE(0)}\langle\psi(0)|T\exp\left\{-i\int_0^t d\tau V(s,\tau)\right\}|\psi(0)\rangle.$$

We now perform an expansion of the overlap

$$\langle\psi(0)|T\exp\left\{-i\int_0^t d\tau V(s,\tau)\right\}|\psi(0)\rangle$$

in powers of $s$. Notice that,

$$T\exp\left\{-i\int_0^t d\tau V(s,\tau)\right\} = I - i\int_0^t d\tau V(s,\tau) - \frac{1}{2}\int_0^t\int_0^t d\tau_2 d\tau_1 T\{V(s,\tau_2)V(s,\tau_1)\} + ...$$

and hence,

$$\frac{d}{ds}\left(T\exp\left\{-i\int_0^t d\tau V(s,\tau)\right\}\right)\Big|_{s=0} = -i\int_0^t d\tau\frac{\partial V}{\partial s}(0,\tau)$$

and,

$$\frac{d^2}{ds^2}\left(T\exp\left\{-i\int_0^t d\tau V(s,\tau)\right\}\right)\Big|_{s=0} = -i\int_0^t d\tau\frac{\partial^2 V}{\partial s^2}(0,\tau) - \int_0^t\int_0^t d\tau_2 d\tau_1 T\left\{\frac{\partial V}{\partial s}(0,\tau_2)\frac{\partial V}{\partial s}(0,\tau_2)\right\}.$$

Therefore,

$$\begin{aligned}
&\langle\psi(0)|T\exp\left\{-i\int_0^t d\tau V(s,\tau)\right\}|\psi(0)\rangle = \\
&1 - is\langle\psi(0)|\int_0^t d\tau\frac{\partial V}{\partial s}(0,\tau)|\psi(0)\rangle \\
&+\frac{s^2}{2}\left[-i\langle\psi(0)|\int_0^t d\tau\frac{\partial^2 V}{\partial s^2}(0,\tau)|\psi(0)\rangle - \langle\psi(0)|\int_0^t\int_0^t d\tau_2 d\tau_1 T\left\{\frac{\partial V}{\partial s}(0,\tau_2)\frac{\partial V}{\partial s}(0,\tau_2)\right\}|\psi(0)\rangle\right] \\
&+\mathrm{O}(s^3)
\end{aligned}$$

Thus, by using the identity $\theta(\tau)+\theta(-\tau)=1$ of the Heaviside theta function, we obtain

$$\begin{aligned}
&|\mathcal{A}(s)|^2 = 1 \\
&- s^2[\int_0^t\int_0^t d\tau_2 d\tau_1(\langle\psi(0)|\frac{1}{2}\left\{\frac{\partial V}{\partial s}(0,\tau_2),\frac{\partial V}{\partial s}(0,\tau_1)\right\}|\psi(0)\rangle \\
&- \langle\psi(0)|\frac{\partial V}{\partial s}(0,\tau_2)|\psi(0)\rangle\langle\psi(0)|\frac{\partial V}{\partial s}(0,\tau_1)|\psi(0)\rangle)] + \mathrm{O}(s^3).
\end{aligned}$$

If we denote the expectation value $\langle\psi(0)|.|\psi(0)\rangle \equiv \langle.\rangle$ we can write,

$$\begin{aligned}
|\mathcal{A}(s)|^2 =& 1 - s^2\int_0^t\int_0^t d\tau_2 d\tau_1\left[\langle\frac{1}{2}\left\{\frac{\partial V}{\partial s}(0,\tau_2),\frac{\partial V}{\partial s}(0,\tau_1)\right\}\rangle - \langle\frac{\partial V}{\partial s}(0,\tau_2)\rangle\langle\frac{\partial V}{\partial s}(0,\tau_1)\rangle\right] \\
&+\mathrm{O}(s^3)
\end{aligned}$$

The quantity,

$$\tilde{\chi} = \int_0^t\int_0^t d\tau_2 d\tau_1\left[\langle\frac{1}{2}\left\{\frac{\partial V}{\partial s}(0,\tau_2),\frac{\partial V}{\partial s}(0,\tau_1)\right\}\rangle - \langle\frac{\partial V}{\partial s}(0,\tau_2)\rangle\langle\frac{\partial V}{\partial s}(0,\tau_1)\rangle\right],$$

is the susceptibility and it is naturally positive. In fact, defining $V_a(\tau) = e^{i\tau H(0)}\partial H/\partial\lambda^a(\lambda_0)e^{-i\tau H(0)}$ such that, by the chain rule,

$$\frac{\partial V}{\partial s}(\tau) = V_a(\tau)\frac{\partial \lambda^a}{\partial s}(0),$$

we can write,

$$\tilde{\chi} = \tilde{g}_{ab}(\lambda_0)\frac{\partial \lambda^a}{\partial s}(0)\frac{\partial \lambda^b}{\partial s}(0),$$

with the metric tensor given by

$$\tilde{g}_{ab}(\lambda_0) = \int_0^t\int_0^t d\tau_2 d\tau_1 \left[\langle\frac{1}{2}\left\{V_a(\tau_2), V_b(\tau_1)\right\}\rangle - \langle V_a(\tau_2)\rangle\langle V_b(\tau_1)\rangle\right]$$

*Loschmidt Echo at finite temperature*

We can naively replace the average $\langle . \rangle \equiv \langle\psi(0)|.|\psi(0)\rangle$ by the average on $\rho(\lambda_0) = \rho(0) = \exp(-\beta H(0))/\mathrm{Tr}\{\exp(-\beta H(0))\}$. Then, the appropriate quantity is the amplitude

$$\mathcal{A}(s) = \mathrm{Tr}\left\{\rho(0)T\exp\left\{-i\int_0^t d\tau V(s,\tau)\right\}\right\}.$$

It is easy to see that $|\mathcal{A}(s)|^2$ has the same expansion as before with the averages replaced by the average on $\rho(0)$.

*Loschmidt Echo susceptibility $\tilde{\chi}$ at finite temperature*

We now proceed to compute $\tilde{\chi}$, or equivalently, $\tilde{g}_{ab}(\lambda_0)$ in the case of a two-level system, where we can write,

$$\rho(\lambda) = \frac{e^{-\beta H(\lambda)}}{\mathrm{Tr}\{e^{-\beta H(\lambda)}\}} = \frac{1}{2}(I - X^\mu(\lambda)\sigma_\mu).$$

and we define variables $r = |X(\lambda)|$ and $n^\mu(\lambda) = X^\mu(\lambda)/|X(\lambda)|$. Writing $H(\lambda) = x^\mu(\lambda)\sigma_\mu$ (and $H(s) \equiv H(\lambda(s))$), we have,

$$V_a(\tau) = \frac{\partial x^\mu}{\partial \lambda^a}(\lambda_0)e^{i\tau H(0)}\sigma_\mu e^{-i\tau H(0)}.$$

Hence,

$$\begin{aligned}\langle V_a(\tau)\rangle &= \frac{1}{\mathrm{Tr}\{e^{-\beta H(\lambda)}\}}\mathrm{Tr}\left\{e^{-\beta H(0)}\sigma_\mu\right\}\frac{\partial x^\mu}{\partial \lambda^a}(\lambda_0)\\ &= r(\lambda_0)n_\mu(\lambda_0)\frac{\partial x^\mu}{\partial \lambda^a}(\lambda_0) = X_\mu(\lambda_0)\frac{\partial x^\mu}{\partial \lambda^a}(\lambda_0),\end{aligned}$$

independent of $\tau$. We then have,

$$\begin{aligned}\langle V_a(\tau_2)\rangle\langle V_b(\tau_1)\rangle &= (r(\lambda_0))^2 n_\mu(\lambda_0)\frac{\partial x^\mu}{\partial \lambda^a}(\lambda_0)n_\nu(\lambda_0)\frac{\partial x^\nu}{\partial \lambda^b}(\lambda_0)\\ &= \tanh^2(\beta|x(\lambda_0)|)\frac{x_\mu}{|x(\lambda_0)|}\frac{\partial x^\mu}{\partial \lambda^a}(\lambda_0)\frac{x_\nu}{|x(\lambda_0)|}\frac{\partial x^\nu}{\partial \lambda^b}(\lambda_0),\end{aligned}$$

where we used $X^\mu(\lambda_0) = \tanh(\beta|x(\lambda_0)|)x^\mu(\lambda_0)/|x(\lambda_0)|$. Now, using the cyclic property of the trace,

$$\begin{aligned}&\frac{1}{2\mathrm{Tr}\{e^{-\beta H(0)}\}}\mathrm{Tr}\left\{e^{-\beta H(\lambda)}\{V_a(\tau_2), V_b(\tau_1)\}\right\} =\\ &\frac{1}{2\mathrm{Tr}\{e^{-\beta H(0)}\}}\mathrm{Tr}\left\{e^{-\beta H(0)}\{\sigma_\mu, \sigma_\nu\}\right\}R^\mu{}_\lambda(\tau_2)R^\nu{}_\sigma(\tau_1)\frac{\partial x^\lambda}{\partial \lambda^a}(\lambda_0)\frac{\partial x^\sigma}{\partial \lambda^b}(\lambda_0)\\ &= \delta_{\mu\nu}R^\mu{}_\lambda(\tau_2)R^\nu{}_\sigma(\tau_1)\frac{\partial x^\lambda}{\partial \lambda^a}(\lambda_0)\frac{\partial x^\sigma}{\partial \lambda^b}(\lambda_0),\end{aligned}$$

where $R^{\mu}{}_{\nu}(\tau)$ is the rotation matrix defined by the equation

$$e^{i\tau H(0)}\sigma_\nu e^{-i\tau H(0)} = R^{\mu}{}_{\nu}(\tau)\sigma_\mu.$$

We can explicitly write it as

$$R^{\mu}{}_{\nu}(\tau) = \cos(2\tau|x(\lambda_0)|)\delta^{\mu}_{\nu} + (1-\cos(2\tau|x(\lambda_0)|))n^{\mu}(\lambda_0)n_\nu(\lambda_0) + \sin(2\tau|x(\lambda_0)|)n^{\lambda}(\lambda_0)\varepsilon_{\lambda}{}^{\mu}{}_{\nu},$$

Using the previous equation and, because $\{R(\tau)\}$ form a one-parameter group,

$$\delta_{\mu\nu}R^{\mu}{}_{\lambda}(\tau_2)R^{\nu}{}_{\sigma}(\tau_1) = \delta_{\kappa\lambda}R^{\kappa}{}_{\sigma}(\tau_2-\tau_1),$$

we find, using the fact that the final expression has to be symmetric under $a \leftrightarrow b$ (hence eliminating unnecessary terms),

$$\begin{aligned}
&\cos\left[2(\tau_2-\tau_1)|x(\lambda_0)|\right]\delta_{\mu\nu}\frac{\partial x^\mu}{\partial\lambda^a}(\lambda_0)\frac{\partial x^\nu}{\partial\lambda^b}(\lambda_0)\\
&+(1-\cos\left[2(\tau_2-\tau_1)|x(\lambda_0)|\right])\frac{x_\mu}{|x(\lambda_0)|}\frac{\partial x^\mu}{\partial\lambda^a}(\lambda_0)\frac{x_\nu}{|x(\lambda_0)|}\frac{\partial x^\nu}{\partial\lambda^b}(\lambda_0).
\end{aligned}$$

Putting everything together,

$$\begin{aligned}
&\langle\frac{1}{2}\{V_a(\tau_2),V_b(\tau_1)\}\rangle - \langle V_a(\tau_2)\rangle\langle V_b(\tau_1)\rangle\\
&=\cos\left[2(\tau_2-\tau_1)|x(\lambda_0)|\right]\langle P\frac{\partial x}{\partial\lambda^a}(\lambda_0), P\frac{\partial x}{\partial\lambda^b}(\lambda_0)\rangle\\
&+(1-\tanh^2(\beta|x(\lambda_0)|))\langle\frac{x(\lambda_0)}{|x(\lambda_0)|},\frac{\partial x}{\partial\lambda^a}(\lambda_0)\rangle\langle\frac{x(\lambda_0)}{|x(\lambda_0)|},\frac{\partial x}{\partial\lambda^b}(\lambda_0)\rangle
\end{aligned}$$

The integral on $\tau_1$ and $\tau_2$ can now be performed using,

$$\begin{aligned}
\int_0^t\int_0^t d\tau_2 d\tau_1\cos[2(\tau_2-\tau_1)\varepsilon] &= \int_0^t\int_0^t d\tau_2 d\tau_1\left(\cos(2\tau_2\epsilon)\cos(2\tau_1\epsilon)+\sin(2\tau_2\epsilon)\sin(2\tau_1\epsilon)\right)\\
&=\frac{1}{4\epsilon^2}\left[\sin^2(2t\epsilon)+(\cos(2t\epsilon)-1)(\cos(2t\epsilon)-1)\right]\\
&=\frac{1}{4\epsilon^2}\left[2-2\cos(2t\epsilon)\right]\\
&=\frac{\sin^2(t\epsilon)}{\epsilon^2}
\end{aligned}$$

So the final result is,

$$\begin{aligned}
\tilde g_{ab}(\lambda_0) = &\frac{\sin^2(|x(\lambda_0)|t)}{|x(\lambda_0)|^2}[\langle P\frac{\partial x}{\partial\lambda^a}(\lambda_0), P\frac{\partial x}{\partial\lambda^b}(\lambda_0)\rangle]\\
&+t^2(1-\tanh^2(\beta|x(\lambda_0|))\langle\frac{x(\lambda_0)}{|x(\lambda_0)|},\frac{\partial x}{\partial\lambda^a}(\lambda_0)\rangle\langle\frac{x(\lambda_0)}{|x(\lambda_0)|},\frac{\partial x}{\partial\lambda^b}(\lambda_0)\rangle,
\end{aligned}$$

*Bures metric for a two-level system*

Take a curve of full-rank density operators $t\mapsto\rho(t)$ and an horizontal lift $t\mapsto W(t)$, with $W(0)=\sqrt{\rho(0)}$. Then the Bures metric is given by

$$g_{\rho(t)}(\frac{d\rho}{dt},\frac{d\rho}{dt}) = \mathrm{Tr}\left\{\frac{dW^\dagger}{dt}\frac{dW}{dt}\right\}.$$

Now the horizontality condition is given by

$$W^\dagger\frac{dW}{dt} = \frac{dW}{dt}^\dagger W,$$

for each $t$. In the full-rank case, we can find a unique Hermitian matrix $G(t)$ such that

$$\frac{dW}{dt} = G(t)W,$$

solves the horizontality condition:

$$W^\dagger \frac{dW}{dt} = W^\dagger G W = \frac{dW}{dt}^\dagger W.$$

Also $G$ is such that

$$\frac{d\rho}{dt} = \frac{d}{dt}(WW^\dagger) = G\rho + \rho G.$$

So if $L_\rho$ ($R_\rho$) is left (right) multiplication by $\rho$, we have, formally,

$$G = (L_\rho + R_\rho)^{-1}\frac{d\rho}{dt}.$$

Therefore,

$$\begin{aligned} g_{\rho(t)}(\frac{d\rho}{dt}, \frac{d\rho}{dt}) &= \mathrm{Tr}\left\{\frac{dW^\dagger}{dt}\frac{dW}{dt}\right\} = \mathrm{Tr}\left\{G^2\rho\right\} \\ &= \frac{1}{2}\mathrm{Tr}\{G(\rho G + G\rho)\} \\ &= \frac{1}{2}\mathrm{Tr}\{G\frac{d\rho}{dt}\} = \frac{1}{2}\mathrm{Tr}\left\{(L_\rho + R_\rho)^{-1}\frac{d\rho}{dt}\frac{d\rho}{dt}\right\}. \end{aligned}$$

Therefore, if we write $\rho(t)$ in the diagonal basis,

$$\rho(t) = \sum_i p_i(t)|i(t)\rangle\langle i(t)|$$

we find,

$$\begin{aligned} g_{\rho(t)}(\frac{d\rho}{dt}, \frac{d\rho}{dt}) &= \frac{1}{2}\mathrm{Tr}\left\{(L_\rho + R_\rho)^{-1}\frac{d\rho}{dt}\frac{d\rho}{dt}\right\} \\ &= \frac{1}{2}\sum_{i,j}\frac{1}{p_i(t) + p_j(t)}\langle i(t)|\frac{d\rho}{dt}|j(t)\rangle\langle j(t)|\frac{d\rho}{dt}|i(t)\rangle. \end{aligned}$$

Hence, we can read off the metric tensor at $\rho$:

$$g_\rho = \frac{1}{2}\sum_{i,j}\frac{1}{p_i + p_j}\langle i|d\rho|j\rangle\langle j|d\rho|i\rangle,$$

using the diagonal basis of $\rho$. This is the result for general full rank density operators. For two-level systems, writing,

$$\rho = \frac{1}{2}(1 - X^\mu\sigma_\mu),$$

and defining variables $|X| = r$ and $n^\mu = X^\mu/|X|$,

$$\begin{aligned} g_\rho &= \left[\frac{1}{1+r} + \frac{1}{1-r}\right]d\rho_{11}^2 + d\rho_{12}d\rho_{21} \\ &= \frac{1}{1-r^2}d\rho_{11}^2 + d\rho_{12}d\rho_{21}, \end{aligned}$$

where we used $d\rho_{11} = -d\rho_{22}$. Notice that,

$$d\rho_{11} = \frac{1}{2}\mathrm{Tr}\{d\rho U\sigma_3 U^{-1}\} = \frac{1}{2}\mathrm{Tr}\{d\rho n^\mu\sigma_\mu\},$$

where $U$ is a unitary matrix diagonalizing $\rho$: $U\sigma_3 U^{-1} = n^\mu \sigma_\mu$. Now,

$$d\rho = -\frac{1}{2} dX^\mu \sigma_\mu,$$

and hence,

$$d\rho_{11} = -\frac{1}{2} dX^\mu n_\mu = -\frac{1}{2} dr.$$

On the other hand,

$$\begin{aligned} d\rho_{12} d\rho_{21} &= \frac{1}{4} \mathrm{Tr}\{d\rho U(\sigma_1 - i\sigma_2)U^{-1}\} \mathrm{Tr}\{d\rho U(\sigma_1 - i\sigma_2)U^{-1}\} \\ &= \frac{1}{4} \left[ \mathrm{Tr}\{d\rho U \sigma_1 U^{-1}\} \mathrm{Tr}\{d\rho U \sigma_1 U^{-1}\} + \mathrm{Tr}\{d\rho U \sigma_2 U^{-1}\} \mathrm{Tr}\{d\rho U \sigma_2 U^{-1}\} \right] \\ &= \frac{1}{4} \delta_{\mu\nu} (dX^\mu - n^\mu n_\lambda dX^\lambda)(dX^\nu - n^\nu n_\sigma dX^\sigma) = \frac{1}{4} r^2 dn^\mu dn_\mu, \end{aligned}$$

where we used the fact that the vectors $(u, v)$ defined by the equations $U\sigma_1 U^{-1} = u^\mu \sigma_\mu$ and $U\sigma_2 U^{-1} = v^\mu \sigma_\mu$ form an orthonormal basis for the orthogonal complement in $\mathbb{R}^3$ of the line generated by $n^\mu$ (which corresponds to the tangent space to the unit sphere $S^2$ at $n^\mu$). The final result is

$$ds^2 = \frac{1}{4} \left( \frac{dr^2}{1 - r^2} + r^2 \delta_{\mu\nu} dn^\mu dn^\nu \right). \tag{14}$$

*Pull-back of the Bures metric*

We have a map

$$M \ni \lambda \mapsto \rho(\lambda) = U(\lambda) \rho_0 U(\lambda)^{-1} = \frac{1}{2} U(\lambda) \left( I - X^\mu \sigma_\mu \right) U(\lambda)^{-1},$$

with

$$U(\lambda) = \exp(-itH(\lambda)),$$

and we take

$$\rho_0 = \frac{\exp(-\beta H(\lambda_0))}{\mathrm{Tr}\left\{ \exp(-\beta H(\lambda_0)) \right\}}, \text{ for some } \lambda_0 \in M.$$

We use the curve $[0,1] \ni s \mapsto \lambda(s)$, with $\lambda(0) = \lambda_0$, to obtain a curve of density operators

$$s \mapsto \rho(s) := \rho(\lambda(s)).$$

Notice that $\rho(0) = \rho_0$. Recall that for $2 \times 2$ density operators of full rank the Bures line element reads

$$ds^2 = \frac{1}{4} \left[ \frac{dr^2}{1 - r^2} + \delta_{\mu\nu} r^2 dn^\mu dn^\nu \right],$$

where

$$n^\mu = X^\mu / |X| \text{ and } r = |X|,$$

with,

$$\rho = \frac{1}{2} \left( I - X^\mu \sigma_\mu \right).$$

Now,

$$\rho(\lambda) = \frac{1}{2} \left( I - R^\mu{}_\nu(\lambda) X^\nu \sigma_\mu \right),$$

with $R^{\mu}{}_{\nu}(\lambda)$ being the unique SO(3) element satisfying

$$U(\lambda)\sigma_\mu U(\lambda)^{-1} = R^{\nu}{}_{\mu}(\lambda)\sigma_\nu.$$

We then have, pulling back the coordinates,

$$r(\lambda) = |X| = \text{constant} \quad \text{and} \quad n^\mu(\lambda) = R^{\mu}{}_{\nu}(\lambda)n^\nu.$$

Therefore,

$$ds^2 = \frac{1}{4}r^2\delta_{\mu\nu}\frac{\partial n^\mu}{\partial\lambda^a}\frac{\partial n^\nu}{\partial\lambda^b}d\lambda^a d\lambda^b = \frac{1}{4}r^2\delta_{\mu\nu}n^\sigma n^\tau\frac{\partial R^{\mu}{}_{\sigma}}{\partial\lambda^a}\frac{\partial R^{\nu}{}_{\tau}}{\partial\lambda^b}d\lambda^a d\lambda^b.$$

in terms of the Euclidean metric on the tangent bundle of $\mathbb{R}^3$, denoted $\langle .,. \rangle$,

$$g_{ab}(\lambda) = \frac{1}{4}r^2\langle R^{-1}\frac{\partial R}{\partial\lambda^a}n, R^{-1}\frac{\partial R}{\partial\lambda^b}n\rangle,$$

written in terms of the pull-back of the Maurer-Cartan form in SO(3), $R^{-1}dR$. We can further pullback by the curve $s \mapsto \lambda(s)$ and evaluate at $s = 0$,

$$\chi := g_{ab}(\lambda_0)\frac{\partial\lambda^a}{\partial s}(0)\frac{\partial\lambda^b}{\partial s}(0) = \frac{1}{4}r^2\langle R^{-1}\frac{\partial R}{\partial\lambda^a}n, R^{-1}\frac{\partial R}{\partial\lambda^b}n\rangle\frac{\partial\lambda^a}{\partial s}(0)\frac{\partial\lambda^b}{\partial s}(0),$$

and this will give us an expansion of the fidelity

$$F(s) \equiv F(\rho(0), \rho(s)) = 1 - \frac{1}{2}\chi s^2 + ...$$

We now evaluate $\chi$. Note that,

$$\begin{aligned} dU\sigma_\mu U^{-1} + U\sigma_\mu dU^{-1} &= dR^{\nu}{}_{\mu}\sigma_\nu \\ U[U^{-1}dU, \sigma_\mu]U^{-1} &= dR^{\nu}{}_{\mu}\sigma_\nu. \end{aligned}$$

or

$$[U^{-1}dU, \sigma] = \sigma \cdot R^{-1}dR.$$

Now, we can parameterise

$$U = y^0 I + iy^\mu\sigma_\mu, \text{ with } |y|^2 = 1.$$

Therefore,

$$\begin{aligned} U^{-1}dU &= (y^0 - iy^\mu\sigma_\mu)(dy^0 + idy^\nu\sigma_\nu) \\ &= i(y^0 dy^\mu - y^\mu dy^0)\sigma_\mu + \frac{i}{2}(y^\mu dy^\nu - y^\nu dy^\mu)\varepsilon_{\mu\nu}{}^{\lambda}\sigma_\lambda. \end{aligned}$$

and,

$$\begin{aligned} [U^{-1}dU, \sigma_\kappa] &= -2\left[(y^0 dy^\mu - y^\mu dy^0)\varepsilon_{\mu\kappa}{}^{\tau} + \frac{1}{2}(y^\mu dy^\nu - y^\nu dy^\mu)\varepsilon_{\mu\nu}{}^{\lambda}\varepsilon_{\lambda\kappa}{}^{\tau}\right]\sigma_\tau \\ &= -2\left[(y^0 dy^\mu - y^\mu dy^0)\varepsilon_{\mu\kappa}{}^{\tau} + \frac{1}{2}(y^\mu dy^\nu - y^\nu dy^\mu)(\delta_{\mu\kappa}\delta^\tau_\nu - \delta^\tau_\mu\delta^\nu_\kappa)\right]\sigma_\tau \\ &= -2\left[(y^0 dy^\mu - y^\mu dy^0)\varepsilon_{\mu\kappa}{}^{\tau} + (y^\kappa dy^\tau - y^\tau dy^\kappa)\right]\sigma_\tau \\ &= 2\left[(y^0 dy^\mu - y^\mu dy^0)\varepsilon_{\mu}{}^{\tau}{}_{\kappa} + (y^\tau dy^\kappa - y^\kappa dy^\tau)\right]\sigma_\tau \equiv (R^{-1}dR)^{\tau}{}_{\kappa}\sigma_\tau. \end{aligned}$$

Observe that for $H(\lambda) = x^\mu(\lambda)\sigma_\mu$, we have,

$$y^0(\lambda) = \cos(|x(\lambda)|t) \text{ and } y^\mu = -\sin(|x(\lambda)|t)\frac{x^\mu(\lambda)}{|x(\lambda)|}.$$

Therefore,

$$dy^0 = -\sin(|x(\lambda)|t)d|x(\lambda)|,$$
$$dy^\mu = -\cos(|x(\lambda)|t)\frac{x^\mu(\lambda)}{|x(\lambda)|}d|x(\lambda)| - \sin(|x(\lambda)|t)d\left(\frac{x^\mu(\lambda)}{|x(\lambda)|}\right).$$

After a bit of algebra, we get,

$$y^0dy^\mu - y^\mu dy^0 = \frac{x^\mu(\lambda)}{|x(\lambda)|}d|x(\lambda)| - \sin(|x(\lambda)|)\cos(|x(\lambda)|)d\left(\frac{x^\mu(\lambda)}{|x(\lambda)|}\right),$$
$$y^\mu dy^\nu - y^\nu dy^\mu = 2\sin^2(|x(\lambda)|t)\frac{x^{[\mu}(\lambda)}{|x(\lambda)|}d\left(\frac{x^{\nu]}(\lambda)}{|x(\lambda)|}\right).$$

So that,

$$(R^{-1}dR)^\tau_\kappa = 2\frac{x^\mu(\lambda)}{|x(\lambda)|}d|x(\lambda)|\varepsilon_\mu{}^\tau{}_\kappa - \sin(2|x(\lambda)|t)d\left(\frac{x^\mu(\lambda)}{|x(\lambda)|}\right)\varepsilon_\mu{}^\tau{}_\kappa + 4\sin^2(|x(\lambda)|t)\frac{x^{[\tau}(\lambda)}{|x(\lambda)|}d\left(\frac{x^{\kappa]}(\lambda)}{|x(\lambda)|}\right).$$

At $s=0$, $\lambda(0)=\lambda_0$ and the coordinate $n^\mu(\lambda(0)) = x^\mu(\lambda_0)/|x(\lambda_0)|$, so that,

$$\begin{aligned}(R^{-1}dR(\lambda_0))^\tau{}_\kappa n^\kappa &= (R^{-1}dR(\lambda_0))^\tau{}_\kappa\frac{x^\kappa(\lambda_0)}{|x(\lambda_0)|}\\ &= -\sin(2|x(\lambda_0)|t)\frac{1}{|x(\lambda_0)|^2}\varepsilon^\tau{}_{\mu\kappa}x^\mu(\lambda_0)dx^\kappa(\lambda_0)\\ &\quad -(1-\cos(2|x(\lambda_0)|t))d\left(\frac{x^\mu}{|x|}\right)(\lambda_0)\end{aligned}$$

Notice that the first term is perpendicular to the second. Therefore, we find,

$$\begin{aligned}\chi ds^2 &= \frac{1}{4}r^2|R^{-1}dRn|^2 = \frac{1}{4}r^2[\sin^2(2|x(\lambda_0)|t)\frac{1}{|x(\lambda_0)|^4}(\delta^\lambda_\mu\delta^\sigma_\kappa - \delta^\sigma_\mu\delta^\lambda_\kappa)x^\mu(\lambda)dx^k(\lambda)x_\lambda(\lambda_0)dx_\sigma(\lambda_0)\\ &\quad+(1-\cos(2|x(\lambda_0|t)^2\langle Pdx(\lambda_0), Pdx(\lambda_0)\rangle)]\\ &= r^2\frac{\sin^2(|x(\lambda_0)|t)}{|x(\lambda_0)|^2}\langle P\frac{\partial x}{\partial\lambda^a}(\lambda_0), P\frac{\partial x}{\partial\lambda^b}(\lambda_0)\rangle\frac{\partial\lambda^a}{\partial s}(0)\frac{\partial\lambda^b}{\partial s}(0)ds^2,\end{aligned}$$

where we have introduced the projector $P: T_x\mathbb{R}^3 = T_xS^2_{|x|}\oplus N_xS^2_{|x|}\to T_xS^2_{|x|}$ onto the tangent space of the sphere of radius $|x|$ at $x$. In other words, the Pullback metric by $\rho$ of the Bures metric at $\lambda_0$ is given by

$$\begin{aligned}g_{ab}(\lambda_0) &= r^2\frac{\sin^2(|x(\lambda_0)|t)}{|x(\lambda_0)|^2}\langle P\frac{\partial x}{\partial\lambda^a}(\lambda_0), P\frac{\partial x}{\partial\lambda^b}(\lambda_0)\rangle\\ &= \tanh^2(\beta|x(\lambda_0)|)\frac{\sin^2(|x(\lambda_0)|t)}{|x(\lambda_0)|^2}\langle P\frac{\partial x}{\partial\lambda^a}(\lambda_0), P\frac{\partial x}{\partial\lambda^b}(\lambda_0)\rangle,\end{aligned}$$

where we noticed that $r=|X|=\tanh(\beta|x(\lambda_0)|)$. The expression $\langle P\frac{\partial x}{\partial\lambda^a}(\lambda_0), P\frac{\partial x}{\partial\lambda^b}(\lambda_0)\rangle$ is just the Riemannian metric on the sphere $S^2_{|x|}\subset\mathbb{R}^3$ evaluated on the tangent vectors $P\frac{\partial x}{\partial\lambda^a}(\lambda_0)$ and $P\frac{\partial x}{\partial\lambda^b}(\lambda_0)$. Notice that at infinite temperature, since the state $\rho(\lambda)$ goes to the totally mixed state $I/2$, the susceptibility is zero. This is to be compared with

$$\begin{aligned}\tilde\chi &= \tilde g_{ab}(\lambda_0)\frac{\partial\lambda^a}{\partial s}(0)\frac{\partial\lambda^b}{\partial s}(0)\\ &= \int_0^t\int_0^t d\tau_2 d\tau_1\left[\langle\frac{1}{2}\left\{\frac{\partial V}{\partial s}(0,\tau_2), \frac{\partial V}{\partial s}(0,\tau_1)\right\}\rangle - \langle\frac{\partial V}{\partial s}(0,\tau_2)\rangle\langle\frac{\partial V}{\partial s}(0,\tau_1)\rangle\right]\\ &= \{\frac{\sin^2(|x(\lambda_0)|t)}{|x(\lambda_0)|^2}[\langle P\frac{\partial x}{\partial\lambda^a}(\lambda_0), P\frac{\partial x}{\partial\lambda^b}(\lambda_0)\rangle]\\ &\quad+t^2(1-\tanh^2(\beta|x(\lambda_0|))\langle\frac{x(\lambda_0)}{|x(\lambda_0)|}, \frac{\partial x}{\partial\lambda^a}(\lambda_0)\rangle\langle\frac{x(\lambda_0)}{|x(\lambda_0)|}, \frac{\partial x}{\partial\lambda^b}(\lambda_0)\rangle\}\frac{\partial\lambda^a}{\partial s}(0)\frac{\partial\lambda^b}{\partial s}(0),\end{aligned}$$

with the average done using the thermal state $\rho_0 = \rho(0) \equiv \rho(\lambda_0)$. This term appears in the expansion of the square of the interferometric amplitude,

$$\mathcal{A}(s) = \mathrm{Tr}\left[\rho(0)\exp(itH(0))\exp(-itH(s))\right] = \mathrm{Tr}\left[\rho(0)T\exp(-i\int_0^t d\tau V(s,\tau))\right],$$

namely,

$$|\mathcal{A}(s)|^2 = 1 - \tilde{\chi}s^2 + ...$$

The difference between the two susceptibilities is given by:

$$\begin{aligned}\tilde{\chi} - \chi = (1-\tanh^2(\beta|x(\lambda_0)|))\{&\frac{\sin^2(|x(\lambda_0)|t)}{|x(\lambda_0)|^2}[\langle P\frac{\partial x}{\partial\lambda^a}(\lambda_0), P\frac{\partial x}{\partial\lambda^b}(\lambda_0)\rangle]\\ &+ t^2\langle\frac{x(\lambda_0)}{|x(\lambda_0)|}, \frac{\partial x}{\partial\lambda^a}(\lambda_0)\rangle\langle\frac{x(\lambda_0)}{|x(\lambda_0)|}, \frac{\partial x}{\partial\lambda^b}(\lambda_0)\rangle\}\frac{\partial\lambda^a}{\partial s}(0)\frac{\partial\lambda^b}{\partial s}(0).\end{aligned}$$

As $\beta \to +\infty$, i.e., as the temperature goes to zero, the two susceptibilities are equal. Now, the function

$$f(t) = \frac{\sin^2(\epsilon t)}{\epsilon^2},$$

when $\epsilon$ is arbitrary small is well approximated by $t^2$. In that case the sum of the two terms appearing in the difference between susceptibilities is just proportional the pull-back Euclidean metric on $T\mathbb{R}^3$.

### The pullback of the inteferometric (Riemannian) metric on the space of unitaries

We first observe that each full rank density operator $\rho$ defines a Hermitian inner product in the vector space of linear maps of a Hilbert space $\mathscr{H}$, i.e., $\mathrm{End}(\mathscr{H})$, given by,

$$\langle A, B\rangle_\rho \equiv \mathrm{Tr}\{\rho A^\dagger B\}.$$

This inner product then defines a Riemannian metric on the trivial tangent bundle of the vector space $\mathrm{End}(\mathscr{H})$. Since the unitary group $\mathrm{U}(\mathscr{H}) \subset \mathrm{End}(\mathscr{H})$, by restriction we get a Riemannian metric on $\mathrm{U}(\mathscr{H})$. If we choose $\rho$ to be $e^{-\beta H(\lambda)}/\mathrm{Tr}\{e^{-\beta H(\lambda)}\}$, then take the pullback by the map $\Phi_t : M \ni \lambda_f \mapsto e^{-itH(\lambda_f)} \in \mathrm{U}(\mathscr{H})$ and evaluate at $\lambda_f = \lambda$, we obtain the desired metric.

Next, we show that this version of LE is closely related to the interferometric geometric phase introduced by Sjöqvist *et. al* [44, 48]. To see this, consider the family of distances in $\mathrm{U}(\mathscr{H})$, $d_\rho$, parametrised by a full rank density operator $\rho$, defined as

$$\begin{aligned}d_\rho^2(U_1, U_2) &= \mathrm{Tr}\{\rho(U_1 - U_2)^\dagger(U_1 - U_2)\}\\ &= 2(1 - \mathrm{Re}\langle U_1, U_2\rangle_\rho),\end{aligned}$$

where $\langle\cdot,\cdot\rangle_\rho$ is the Hermitian inner product defined previously. In terms of the spectral representation of $\rho = \sum_j p_j|j\rangle\langle j|$, we have

$$\langle U_1, U_2\rangle_\rho = \sum_j p_j\langle j|U_1^\dagger U_2|j\rangle.$$

The Hermitian inner product is invariant under $U_i \mapsto U_i \cdot D$, $i = 1, 2$, where $D$ is a phase matrix

$$D = e^{i\alpha}\sum_j |j\rangle\langle j|.$$

For the interferometric geometric phase, one enlarges this gauge symmetry to the subgroup of unitaries preserving $\rho$. However, since we are interested in the interferometric LE previously defined, we choose not to do that. Next, promoting this global U(1)-gauge degree of freedom to a local one, i.e., demanding that we only care about unitaries modulo a phase, we see that, upon changing $U_i \mapsto U_i \cdot D_i$, $i = 1, 2$, we have,

$$\begin{aligned}\langle U_1, U_2\rangle_\rho \mapsto & \langle U_1 \cdot D_1, U_2 \cdot D_2\rangle_\rho \\ &= \sum_j p_j \langle j|U_1^\dagger U_2|j\rangle e^{i(\alpha_2-\alpha_1)}.\end{aligned}$$

We can choose gauges, i.e., $D_1$ and $D_2$, minimizing $d_\rho^2(U_1 \cdot D_1, U_2 \cdot D_2)$, obtaining

$$\begin{aligned}d_\rho^2(U_1 \cdot D_1, U_2 \cdot D_2) &= 2(1 - |\langle U_1 \cdot D_1, U_2 \cdot D_2\rangle_\rho|) \\ &= 2(1 - |\langle U_1, U_2\rangle_\rho|).\end{aligned}$$

Now, if $\{U_i = U(t_i)\}_{1\leq i\leq N}$ were the discretisation of a path of unitaries $t \mapsto U(t)$, $t \in [0, 1]$, applying the minimisation process locally, i.e., between adjacent unitaries $U_{i+1}$ and $U_i$, in the limit $N \to \infty$ we get a notion of parallel transport on the principal bundle $\mathrm{U}(\mathscr{H}) \to \mathrm{U}(\mathscr{H})/\mathrm{U}(1)$. In particular, the parallel transport condition reads,

$$\mathrm{Tr}\left\{\rho U^\dagger(t)\frac{dU}{dt}(t)\right\} = 0, \text{ for all } t \in [0, 1].$$

If we take $\rho = \exp(-\beta H(\lambda_i))/\mathrm{Tr}\{e^{-\beta H(\lambda_i)}\}$, $U_1 = \exp(-itH(\lambda_i))$ and $U_2 = \exp(-itH(\lambda_f))$, then the interferometric LE is

$$\mathcal{L}(t, \beta; \lambda_f, \lambda_i) = |\langle U_1, U_2\rangle_\rho| = \langle \widetilde{U}_1, \widetilde{U}_2\rangle_\rho,$$

where $\widetilde{U}_i = U_i \cdot D_i$ $(i = 1, 2)$ correspond to representatives satisfying the discrete version of the parallel transport condition.


[1] Subir Sachdev. *Quantum phase transitions*. Wiley Online Library, 2007.

[2] Lev Davidovich Landau et al. On the theory of phase transitions. *Zh. eksp. teor. Fiz*, 7(19-32), 1937.

[3] Xiao-Gang Wen. Topological orders in rigid states. *International Journal of Modern Physics B*, 4(02):239–271, 1990.

[4] JM Kosterlitz and DJ Thouless. Long range order and metastability in two dimensional solids and superfluids.(application of dislocation theory). *Journal of Physics C: Solid State Physics*, 5(11):L124, 1972.

[5] John Michael Kosterlitz and David James Thouless. Ordering, metastability and phase transitions in two-dimensional systems. *Journal of Physics C: Solid State Physics*, 6(7):1181, 1973.

[6] F Duncan M Haldane. Continuum dynamics of the 1-d heisenberg antiferromagnet: identification with the o (3) nonlinear sigma model. *Physics Letters A*, 93(9):464–468, 1983.

[7] F. D. M. Haldane. Model for a quantum hall effect without landau levels: Condensed-matter realization of the "parity anomaly". *Phys. Rev. Lett.*, 61:2015–2018, Oct 1988.

[8] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs. Quantized hall conductance in a two-dimensional periodic potential. *Phys. Rev. Lett.*, 49:405–408, Aug 1982.

[9] M. Z. Hasan and C. L. Kane. Colloquium: Topological insulators. *Rev. Mod. Phys.*, 82:3045–3067, Nov 2010.

[10] Xiao-Liang Qi and Shou-Cheng Zhang. Topological insulators and superconductors. *Reviews of Modern Physics*, 83(4):1057, 2011.

[11] B Andrei Bernevig and Taylor L Hughes. *Topological insulators and topological superconductors*. Princeton University Press, 2013.

[12] Charles-Edouard Bardyn, Lukas Wawer, Alexander Altland, Michael Fleischhauer, and Sebastian Diehl. Probing the topology of density matrices. *arXiv preprint arXiv:1706.02741*, 2017.

[13] F Grusdt. Topological order of mixed states in correlated quantum many-body systems. *Physical Review B*, 95(7):075106, 2017.

[14] Bruno Mera, Chrysoula Vlachou, Nikola Paunković, and Vítor R. Vieira. Uhlmann connection in fermionic systems undergoing phase transitions. *Phys. Rev. Lett.*, 119:015702, Jul 2017.

[15] Bruno Mera, Chrysoula Vlachou, Nikola Paunković, and Vítor R. Vieira. Boltzmann-gibbs states in topological quantum walks and associated many-body systems: fidelity and uhlmann parallel transport analysis of phase transitions. *J. Phys. A: Math. Theor.*, 50:365302, Aug 2017.

[16] Ole Andersson, Ingemar Bengtsson, Marie Ericsson, and Erik Sjöqvist. Geometric phases for mixed states of the kitaev chain. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 374(2068):20150231, 2016.

[17] O Viyuela, A Rivas, and MA Martin-Delgado. Symmetry-protected topological phases at finite temperature. *2D Materials*, 2(3):034006, 2015.

[18] Evert PL van Nieuwenburg and Sebastian D Huber.

Classification of mixed-state topology in one dimension. *Physical Review B*, 90(7):075141, 2014.
[19] O. Viyuela, A. Rivas, and M. A. Martin-Delgado. Uhlmann phase as a topological measure for one-dimensional fermion systems. *Phys. Rev. Lett.*, 112:130401, Apr 2014.
[20] O. Viyuela, A. Rivas, and M. A. Martin-Delgado. Two-dimensional density-matrix topological fermionic phases: Topological uhlmann numbers. *Phys. Rev. Lett.*, 113:076408, Aug 2014.
[21] CE Bardyn, MA Baranov, CV Kraus, E Rico, A İmamoğlu, P Zoller, and S Diehl. Topology by dissipation. *New Journal of Physics*, 15(8):085001, 2013.
[22] A. Rivas, O. Viyuela, and M. A. Martin-Delgado. Density-matrix chern insulators: Finite-temperature generalization of topological insulators. *Phys. Rev. B*, 88:155141, Oct 2013.
[23] O. Viyuela, A. Rivas, and M. A. Martin-Delgado. Thermal instability of protected end states in a one-dimensional topological insulator. *Phys. Rev. B*, 86:155140, Oct 2012.
[24] M. Heyl, A. Polkovnikov, and S. Kehrein. Dynamical quantum phase transitions in the transverse-field ising model. *Phys. Rev. Lett.*, 110:135704, Mar 2013.
[25] Szabolcs Vajna and Balázs Dóra. Disentangling dynamical phase transitions from equilibrium phase transitions. *Phys. Rev. B*, 89:161105, Apr 2014.
[26] Markus Heyl. Scaling and universality at dynamical quantum phase transitions. *Phys. Rev. Lett.*, 115:140602, Oct 2015.
[27] Szabolcs Vajna and Balázs Dóra. Topological classification of dynamical phase transitions. *Phys. Rev. B*, 91:155127, Apr 2015.
[28] M. Schmitt and S. Kehrein. Dynamical quantum phase transitions in the kitaev honeycomb model. *Phys. Rev. B*, 92:075114, Aug 2015.
[29] J. C. Budich and M. Heyl. Dynamical topological order parameters far from equilibrium. *Phys. Rev. B*, 93:085416, Feb 2016.
[30] Z. Huang and A. V. Balatsky. Dynamical quantum phase transitions: Role of topological nodes in wave function overlaps. *Phys. Rev. Lett.*, 117:086802, Aug 2016.
[31] H. T. Quan, Z. Song, X. F. Liu, P. Zanardi, and C. P. Sun. Decay of loschmidt echo enhanced by quantum criticality. *Phys. Rev. Lett.*, 96:140604, Apr 2006.
[32] Paolo Zanardi and Nikola Paunković. Ground state overlap and quantum phase transitions. *Phys. Rev. E*, 74:031123, Sep 2006.
[33] P. Zanardi, H. T. Quan, X. Wang, and C. P. Sun. Mixed-state fidelity and quantum criticality at finite temperature. *Phys. Rev. A*, 75:032109, Mar 2007.
[34] F. Pollmann, S. Mukerjee, A. G. Green, and J. E. Moore. Dynamics after a sweep through a quantum critical point. *Phys. Rev. E*, 81:020101, Feb 2010.
[35] N. T. Jacobson, L. C. Venuti, and P. Zanardi. Unitary equilibration after a quantum quench of a thermal state. *Phys. Rev. A*, 84:022115, Aug 2011.
[36] L. Campos Venuti and P. Zanardi. Unitary equilibrations: Probability distribution of the loschmidt echo. *Phys. Rev. A*, 81:022113, Feb 2010.
[37] Lorenzo Campos Venuti, N. Tobias Jacobson, Siddhartha Santra, and Paolo Zanardi. Exact infinite-time statistics of the loschmidt echo for a quantum quench. *Phys. Rev. Lett.*, 107:010403, Jul 2011.
[38] M. Heyl and J. C. Budich. Dynamical topological quantum phase transitions for mixed states. *Phys. Rev. B*, 96:180304, Nov 2017.
[39] Utso Bhattacharya, Souvik Bandyopadhyay, and Amit Dutta. Mixed state dynamical quantum phase transitions. *Phys. Rev. B*, 96:180303, Nov 2017.
[40] N. Paunković, P. D. Sacramento, P. Nogueira, V. R. Vieira, and V. K. Dugaev. Fidelity between partial states as a signature of quantum phase transitions. *Phys. Rev. A*, 77:052302, May 2008.
[41] Paolo Zanardi, Lorenzo Campos Venuti, and Paolo Giorda. Bures metric over thermal state manifolds and quantum criticality. *Phys. Rev. A*, 76:062318, Dec 2007.
[42] Damian F. Abasto, Alioscia Hamma, and Paolo Zanardi. Fidelity analysis of topological quantum phase transitions. *Phys. Rev. A*, 78:010301, Jul 2008.
[43] Jian-Hui Zhao and Huan-Qiang Zhou. Singularities in ground-state fidelity and quantum phase transitions for the kitaev model. *Phys. Rev. B*, 80:014403, Jul 2009.
[44] Erik Sjöqvist, Arun K. Pati, Artur Ekert, Jeeva S. Anandan, Marie Ericsson, Daniel K. L. Oi, and Vlatko Vedral. Geometric phases for mixed states in interferometry. *Phys. Rev. Lett.*, 85:2845–2849, Oct 2000.
[45] X.-L. Qi, L. Hughes, T., and S.-C. Zhang. Topological field theory of time-reversal invariant insulators. *Phys. Rev. B*, 78:195424, Nov 2008.
[46] Dariusz Chruscinski and Andrzej Jamiolkowski. *Geometric phases in classical and quantum mechanics*, volume 36. Springer Science & Business Media, 2012.
[47] Armin Uhlmann. Transition probability (fidelity) and its relatives. *Foundations of Physics*, 41(3):288–298, 2011.
[48] D. M. Tong, E. Sjöqvist, L. C. Kwek, and C. H. Oh. Kinematic approach to the mixed state geometric phase in nonunitary evolution. *Phys. Rev. Lett.*, 93:080405, Aug 2004.
[49] C. N. Yang and T. D. Lee. Statistical theory of equations of state and phase transitions. i. theory of condensation. *Phys. Rev.*, 87:404–409, Aug 1952.
[50] T. D. Lee and C. N. Yang. Statistical theory of equations of state and phase transitions. ii. lattice gas and ising model. *Phys. Rev.*, 87:410–419, Aug 1952.
[51] Michael E Fisher. Lectures in theoretical physics. *New York: Gordon and Breach*, 1965.
[52] R. K. Pathria. *Statistical Mechanics*. Butterworth-Heinemann, 1996.
[53] Angelo Bassi, Kinjalk Lochan, Seema Satin, Tejinder P. Singh, and Hendrik Ulbricht. Models of wave-function collapse, underlying theories, and experimental tests. *Rev. Mod. Phys.*, 85:471–527, Apr 2013.
[54] U. Bhattacharya, S. Bandopadhyay, and A. Dutta. Mixed state dynamical quantum phase transition and emergent topology. *arxiv.org/abs/1705.04555*, 2017.
[55] W. P. Su, J. R. Schrieffer, and A. J. Heeger. Solitons in polyacetylene. *Phys. Rev. Lett.*, 42:1698–1701, Jun 1979.
[56] M. J. Rice and E. J. Mele. Elementary excitations of a linearly conjugated diatomic polymer. *Phys. Rev. Lett.*, 49:1455–1459, Nov 1982.
[57] SN Kempkes, A Quelle, and C Morais Smith. Universalities of thermodynamic signatures in topological phases. *Scientific reports*, 6:38530, December 2016.
[58] N. Sedlmayr, M. Fleischhauer, and J. Sirker. The fate of dynamical phase transitions at finite temperatures and in open systems. *ArXiv e-prints*, November 2017.