Dynamics of magnetic moments coupled to electrons and lattice oscillations

B. Mera,[1,2,*] V. R. Vieira,[2] and V. K. Dugaev[2,3,4]

[1]*Physics of Information Group, Instituto de Telecomunicações, 1049-001 Lisbon, Portugal*

[2]*Centro de Física das Interacções Fundamentais, Instituto Superior Técnico, Universidade Técnica de Lisboa, Avenida Rovisco Pais 1, 1049-001 Lisboa, Portugal*

[3]*Department of Physics, Rzeszów University of Technology, aleja Powstańców Warszawy 6, 35-959 Rzeszów, Poland*

[4]*Institut für Physik, Martin-Luther-Universität Halle-Wittenberg, Heinrich-Damerow-Strasse 4, 06120 Halle, Germany*

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Inspired by the models of Rebei and Parker [Phys. Rev. B **67**, 104434 (2003)] and [Rebei, Hitchon and Parker Phys. Rev. B **72**, 064408 (2005)], we study a physical model which describes the behavior of magnetic moments in a ferromagnet. The magnetic moments are associated to $3d$ electrons which interact with conduction-band electrons and with phonons. We study each interaction separately and then collect the results, assuming that the electron-phonon interaction can be neglected. For the case of the spin-phonon interaction, we study the derivation of the equations of motion for the classical spin vector and find that the correct behavior, as given by the Brown equation for the spin vector and the Bloch equation, using the results obtained by Garanin [Phys. Rev. B **55**, 3050 (1997)] for the average over fluctuations of the spin vector, can be obtained in the high-temperature limit. At finite temperatures, we show that the Markovian approximation for the fluctuations is not correct for time scales below some thermal correlation time τ_{th}. For the case of electrons we work a perturbative expansion of the Feynman-Vernon influence functional. We find the expression for the random field correlation function which exhibits the properties of the electron bath; namely, we observe Friedel oscillations at small temperatures. The composite model (as well as the individual models) is shown to satisfy a fluctuation-dissipation theorem for all temperature regimes if the behavior of the coupling constants of the phonon-spin interaction remain unchanged with the temperature. The equations of motion are derived.

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I. INTRODUCTION

The discovery of the Giant Magnetoresistance effect in 1988, for which Grünberg and Fert were awarded the Nobel Prize in Physics in 2007, motivated very intense scientific research of the magnetization dynamics at the nanometer scale and led to the birth of a new field of research now called spintronics.[1–3] Spintronic devices usually have nanometer-scale sizes, can operate at high frequencies ($\sim$1 GHz), and have a wide range of applications which go from the creation of small-dimension ($<$1 μm) microwave frequency generators to the improvement of magnetic storage devices.

To successfully design high-frequency devices one needs to develop substantially the theoretical comprehension of magnetization dynamics at the appropriate scales.[4–6] The complete understanding of magnetization dynamics at the nanoscale can be probably achieved only by theorizing from first principles implying a full quantum-mechanical treatment. In particular, if one wants to describe a spin system far from equilibrium, one needs to use the methods of quantum open systems far from equilibrium, namely the Keldysh[7] or Lindblad[8] formalisms.

It has been shown[9] that the linear coupling interaction of a spin with a bosonic bath allows for the existence of white noise in the equation of motion which, under some particular conditions regarding the density of states of the bath, adopts the form of the Landau-Lifshitz-Gilbert-Brown equation.[10–12] Also, it has been shown that if the spin vector satisfies a Landau-Lifshitz equation[10] supplemented with white noise, then the magnetic moment as the average over the fluctuations of the spin satisfies, in the limit of low temperatures, the Landau-Lifshitz equation[10] and, in the limit of high temperatures, the Landau-Lifshitz-Bloch[13] equation.[14] Collecting these results with the known result on formal equivalence between Landau-Lifshitz and Landau-Lifshitz-Gilbert equations allows to conclude that the interaction with phonons (or any other bosonic bath satisfying certain conditions) can be responsible for the motion of magnetic moments as described by the Landau-Lifshitz-Bloch equation, which, in fact, gives a good description of the physical situation at high temperatures.[15] A quantum field theoretical treatment of the s-d interaction of conduction electrons and spins[16] has shown that in the semiclassical limit, the magnetization obeys a generalized Landau-Lifshitz equation.

The need to increase the speed of storage of information in magnetic media and the limitations associated with the generation of magnetic field pulses by an electric current require the research for ways of controlling magnetization by other means than external magnetic fields. In 1996, subpicosecond demagnetization in ferromagnetic nickel was achieved using a 60-fs laser in the experiments of Beaurepaire *et al.*[17] Manipulating magnetization with ultrashort (of the order of a femtosecond) laser pulses is now a major research challenge because at such time scales it might be possible to reverse the magnetization faster than within half a precessional period.[18] Because of this, it is of fundamental importance to understand the time evolution of magnetic moments at high temperatures and time scales approaching femtoseconds. Recent reviews on the state of the art of ultrafast spin dynamics and prospects are given, for example, by Kyrilyuk *et al.*[18] and Zhang *et al.*[19]

The ultrashort laser pulses are expected to strongly couple lattice oscillations, that is, phonons, and/or conduction electrons to the spins. This suggests that we may consider an effective microscopic theory for the system in which the fundamental interactions are spin-phonon, spin-electron, and electron-phonon type. In this work we introduce and develop a theory to model such physical system in which, for the sake of simplicity, we neglect the electron-phonon interaction.

This paper is divided into three major sections. In the first section we consider a generalized version of the model of Rebei and Parker[9] that modeled the interaction of a spin-j and a bosonic bath, now written to describe the interaction between a spin-j and a bath of phonons which are assumed to be spin 1. The second section considers a model of a ferromagnet, inspired by the work of Rebei *et al.*,[16] assuming the interaction of a magnetic system of $3d$ electrons, represented by a collection of spin-j vectors, with $4s$ conduction electrons as a bath. In both sections we use the path-integral representation for coherent-state matrix elements of the reduced density matrix associated with the system. In the case of phonons, since we assume a linear interaction, they can be exactly integrated out. After obtaining the effective action appearing in the phase of the path integral we use a stationary phase approximation and obtain an equation of motion for the classical spin vector. We then display a random field in the equations of motion through a Hubbard-Stratonovich transformation in the path-integral expression. The high-temperature limit is discussed and the Landau-Lifshitz-Gilbert-Brown and Landau-Lifshitz-Bloch equations are recovered in this limit. The finite-temperature case is considered.

In the case of conduction electrons interacting with spins, the same procedure of integrating out the bath is not possible due to the nonlinearity of the interaction. Nevertheless, we use a Hubbard-Stratonovich transformation and obtain an expansion for the effective action of the system which we truncate at the second order of interaction. We relate this approximation to the Lindblad equation[8] for the reduced density matrix obtained through the Born-Markov approximation.[20] After that we arrive at an explicit expression of the electron contribution to the bath noise correlation function and present some plots of this function which manifest the physical phenomena involved.

The third section compiles the results of the first and second sections in a single model assuming a quantum system composed of a spin-j vector field interacting with a bath of spin-$\frac{1}{2}$ electrons and spin-1 phonons. Finally, we list our conclusions.

II. SPIN-PHONON THEORY

We consider first the system of a single spin-j particle interacting with a bath of spin-1 phonons which can have longitudinal and transverse polarizations.

The total Hamiltonian is written in the form

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_S + \hat{\mathcal{H}}_R + \hat{\mathcal{H}}_i\,, \tag{1}$$

presenting, respectively, the Hamiltonians for spin, bath, and interaction between them.

We can introduce quantum operators associated to the spin-j particle $\hat{S}_\alpha$ which are Hermitian operators satisfying the angular momentum commutation relations

$$[\hat{S}_\alpha,\hat{S}_\beta] = i\varepsilon_{\alpha\beta\gamma}\hat{S}_\gamma, \tag{2}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Civita tensor, and the repeated indices are assumed to be summed from now on, unless otherwise stated. These operators constitute a representation of the Lie algebra of SU(2) in the $(2j+1)$-dimensional vector space associated to the irreducible representation labeled by j. We also introduce the creation and annihilation operators for the phonons, $\hat{a}^\dagger_\mu(\mathbf{k})$, $\hat{a}_\mu(\mathbf{k})$, which are labeled by the polarization index $\mu = -1$, 0, or 1 and by the momentum index $\mathbf{k}$. They satisfy the Weyl algebra

$$[\hat{a}_\mu(\mathbf{k}),\hat{a}^\dagger_\nu(\mathbf{k}')] = \delta_{\mu\nu}(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}'). \tag{3}$$

The spin operators and the phonon operators commute.

The system Hamiltonian is that of a spin-j particle in the presence of an external magnetic field, denoted by $\mathbf{H}$,

$$\hat{\mathcal{H}}_S = -\hat{S}_\alpha H_\alpha = -\hat{\mathbf{S}}\cdot\mathbf{H}. \tag{4}$$

We write the phonon Hamiltonian in the form

$$\hat{\mathcal{H}}_R = \sum_{\mu,\mathbf{k}} \omega_\mu(\mathbf{k})\hat{a}^\dagger_\mu(\mathbf{k})\hat{a}_\mu(\mathbf{k}), \tag{5}$$

where $\omega_\mu(\mathbf{k})$ is the phonon energy. In the following we assume that transverse modes have the same energy, $\omega_{-1}(\mathbf{k}) = \omega_1(\mathbf{k})$.

The Hermitian Hamiltonian for the linear spin-phonon coupling in the macrospin approximation[21] is

$$\hat{\mathcal{H}}_i = -\sum_{\mathbf{k}}(H^*_{\alpha\mu}(\mathbf{k})\hat{a}^\dagger_\mu(\mathbf{k})\hat{S}_\alpha + H_{\alpha\mu}(\mathbf{k})\hat{S}_\alpha\hat{a}_\mu(\mathbf{k})). \tag{6}$$

Here the SU(2) invariance is explicit provided that the operator

$$\hat{H}^{\rm ph}_\alpha := \sum_{\mathbf{k}}(H^*_{\alpha\mu}(\mathbf{k})\hat{a}^\dagger_\mu(\mathbf{k}) + H_{\alpha\mu}(\mathbf{k})\hat{a}_\mu(\mathbf{k})) \tag{7}$$

transforms as a vector under an SU(2) transformation associated with the spin-j particle and consequently $\hat{\mathcal{H}}_i$ will behave as a scalar. We can then write the interaction Hamiltonian as

$$\hat{\mathcal{H}}_i = -\hat{S}_\alpha\hat{H}^{\rm ph}_\alpha = -\hat{\mathbf{S}}\cdot\hat{\mathbf{H}}^{\rm ph}. \tag{8}$$

Writing the system reduced density matrix with a coherent-state path-integral representation, one arrives, under the Keldysh formalism, at an effective action [see Eq. (A30)] in the closed time path which is a functional of $\mathbf{S}$, the classical field, and $\mathbf{D}$, associated to the quantum fluctuations. The details of this calculation are presented in Appendix A. Performing a Hubbard-Stratonovich transformation on the part of the effective action which couples two $\mathbf{D}(t)$ fields, as in Ref. 9, we obtain the equations of motion with a random field $\boldsymbol{\xi}(t)$. The real part of the correlation function of this random field is

$$\begin{aligned}\mathrm{Re}\{\langle\xi_\alpha(t_1)\xi_\beta(t_2)\rangle\} &= \mathrm{Re}\{\langle\xi_\alpha(0)\xi_\beta(t_1-t_2)\rangle\}\\ &= \mathrm{Re}\Bigg\{\sum_{\mathbf{k},\mu} H^*_{\alpha\mu}(\mathbf{k})e^{-i\omega_\mu(\mathbf{k})(t_1-t_2)}\\ &\quad\times(1+2n(\omega_\mu(\mathbf{k})))H_{\beta\mu}(\mathbf{k})\Bigg\}. \end{aligned}\tag{9}$$

The last expression can be rewritten as

$$\mathrm{Re}\left\{\sum_{\mathbf{k},\mu}\int_0^\infty d\omega\delta(\omega-\omega_\mu(\mathbf{k}))H^*_{\alpha\mu}(\mathbf{k})H_{\beta\mu}(\mathbf{k}) \times e^{-i\omega_\mu(\mathbf{k})(t_1-t_2)}\coth\left(\frac{\beta\omega_\mu(\mathbf{k})}{2}\right)\right\}. \tag{10}$$

Let us assume that $H_{\alpha\mu}(\mathbf{k}) = H_\alpha(\omega_\mu(\mathbf{k}))$. In this case, we can further write, interchanging the sum and integral and using the property $f(\omega_k)\delta(\omega_k-\omega) = f(\omega)\delta(\omega_k-\omega)$ of the δ distribution,

$$\mathrm{Re}\left\{\int_0^\infty d\omega\sum_{\mathbf{k},\mu}\delta(\omega-\omega_\mu(\mathbf{k}))H^*_\alpha(\omega)H_\beta(\omega)e^{-i\omega(t_1-t_2)} \times\coth\left(\frac{\beta\omega}{2}\right)\right\} \tag{11}$$

or, equivalently,

$$\mathrm{Re}\left\{\int_0^\infty d\omega\,\rho(\omega)H^*_\alpha(\omega)H_\beta(\omega)e^{-i\omega(t_1-t_2)} \times\coth\left(\frac{\beta\omega}{2}\right)\right\}, \tag{12}$$

where we defined the density of states

$$\rho(\omega) = \sum_{\mathbf{k},\mu}\delta(\omega-\omega_\mu(\mathbf{k})). \tag{13}$$

In the high-temperature limit, the expression (12) reads

$$2k_BT\,\mathrm{Re}\left\{\int_0^\infty\rho(\omega)\frac{H^*_\alpha(\omega)H_\beta(\omega)}{\omega}e^{-i\omega(t_1-t_2)}d\omega\right\}. \tag{14}$$

Assuming a linear dispersion relation for longitudinal and transverse phonons, we obtain

$$\omega_0(\mathbf{k}) = c_lk, \quad \omega_{\pm1}(\mathbf{k}) = c_tk, \tag{15}$$

and if we take the continuum limit for the bath, we arrive at the density of states

$$\begin{aligned}\rho(\omega) &= \sum_{\mathbf{k},\mu}\delta(\omega-\omega_\mu(\mathbf{k}))\\ &= V\sum_\mu\int\frac{d^3k}{(2\pi)^3}\delta(\omega-\omega_\mu(\mathbf{k}))\\ &= \frac{V}{2\pi^2}\left(\frac{1}{c_l^3}+\frac{2}{c_t^3}\right)\omega^2 = \rho_0\left(\frac{\omega}{\omega_0}\right)^2,\end{aligned} \tag{16}$$

where V is the volume of the reservoir and $\rho_0 = \rho(\omega_0)$ is the density of states evaluated at some frequency ω_0 taken as a reference. This means that if we want to recover the Gaussian random field in the limit of high temperatures we must ensure that

$$\mathrm{Re}\{H^*_\alpha(\omega)H_\beta(\omega)\} \propto \frac{1}{\omega}. \tag{17}$$

In particular, one must have

$$\frac{\mathrm{Re}\{H^*_\alpha(\omega)H_\beta(\omega)\}\rho(\omega)\pi}{\omega} = \alpha_G\delta_{\alpha\beta}, \tag{18}$$

so that

$$\mathrm{Re}\{\langle\xi_\alpha(t)\xi_\beta(t')\rangle\} = \delta_{\alpha\beta}2\alpha_Gk_BT\delta(t-t'). \tag{19}$$

The constant α_G is a parameter which gives the intensity of the Gilbert damping term in the equation of motion. The last condition constrains the form of $H_\alpha(\omega)$. If we define

$$z_\alpha = \left(\frac{\pi\rho_0}{\alpha_G\omega_0^2}\right)^{1/2}\omega^{1/2}H_\alpha(\omega), \tag{20}$$

then the above condition reads

$$z^*_\alpha z_\beta + z^*_\beta z_\alpha = 2\delta_{\alpha\beta}, \tag{21}$$

which means that

$$z_\alpha = e^{i\varphi_\alpha} \tag{22}$$

and that the phases must satisfy

$$\varphi_\alpha = \varphi_\beta + (2n+1)\frac{\pi}{2} \tag{23}$$

for all α different from β and with n being an integer. The form of $H_\alpha(\omega)$ is, thus, constrained to be

$$H_\alpha(\omega) = \left(\frac{\pi\rho_0}{\alpha_G\omega_0^2}\right)^{-1/2}\omega^{-1/2}e^{i\varphi_\alpha} \tag{24}$$

if the familiar fluctuation-dissipation theorem is satisfied.

Assuming the above dependence for $H_\alpha(\omega)$ for arbitrary temperatures, we find that the noise correlation function can be written as

$$\begin{aligned}\mathrm{Re}\{\langle\xi_\alpha(t)\xi_\beta(0)\rangle\} &= \alpha_G\delta_{\alpha\beta}\int_0^\infty\frac{d\omega}{\pi}\,\omega\coth\left(\frac{\beta\omega}{2}\right)\cos\omega t\\ &= \alpha_G\delta_{\alpha\beta}\varphi(t),\end{aligned} \tag{25}$$

where we define the generalized function φ:

$$\varphi(t) = \int_0^\infty\frac{d\omega}{\pi}\,\omega\coth\left(\frac{\beta\omega}{2}\right)\cos\omega t, \tag{26}$$

which characterizes the noise correlation function. The study of such a generalized function or distribution can be found, for instance, in Ref. 22.

To study the random field correlation function we notice that the above integral can be expressed as

$$\begin{aligned}\varphi(t) &= \int_0^\infty\frac{d\omega}{\pi}\left[\coth\left(\frac{\beta\omega}{2}\right)\omega-\omega\right]\cos\omega t\\ &\quad+\frac{1}{2}\left(\int_{-\infty}^\infty\frac{d\omega}{2\pi}\,|\omega|e^{-i\omega t}\right).\end{aligned} \tag{27}$$

Here the first integral is convergent and can be computed as

$$\begin{aligned}&\int_0^\infty\frac{d\omega}{\pi}\left[\coth\left(\frac{\beta\omega}{2}\right)\omega-\omega\right]\cos\omega t\\ &= \frac{1}{2\pi}\left\{\frac{1}{t^2}-\left(\frac{\pi}{\beta}\right)^2\mathrm{cosech}^2\left[\left(\frac{\pi}{\beta}\right)t\right]\right\}.\end{aligned} \tag{28}$$

The second integral can be computed if we regularize it with an exponential cutoff,

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\omega| e^{-i\omega t} \to \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\omega| e^{-i\omega t} e^{-\lambda\omega} = \frac{1}{\pi} \frac{\lambda^2 - t^2}{(\lambda^2 + t^2)^2}$$
$$= \frac{1}{\pi} \left(-\frac{1}{t^2} + \frac{\lambda^2(\lambda^2 + 3t^2)}{t^2(\lambda^2 + t^2)^2} \right), \ \lambda \to 0^+. \tag{29}$$

Thus, formally, we can write

$$\varphi(t) = -\frac{\pi}{2\beta^2} \text{cosech}^2 \left[\left(\frac{\pi}{\beta} \right) t \right], \tag{30}$$

which is a well-known result.[23] The correlation exhibits a thermal correlation time arising from the behavior of φ in the limit of $|t| \to \infty$,

$$\varphi(t) \to -2\pi (k_B T)^2 \exp(-2\pi k_B T |t|)$$
$$= -2\pi (k_B T)^2 \exp\left(-\frac{|t|}{\tau_{\text{th}}} \right), \tag{31}$$

where $\tau_{\text{th}} = (2\pi k_B T)^{-1}$. In the case of arbitrarily small temperatures, the correlation time becomes very large [in fact, as seen from Eq. (30) the correlation function becomes inverse quadratic in $|t|$].

It is important to note that the long time behavior of the correlations is negative. One must study the full expression for φ for finite cutoff λ to understand what is happening in this limit. We have

$$\int_{-\infty}^{\infty} dt \ \text{Re}\{\langle \xi_\alpha(t) \xi_\beta(0) \rangle\} = 2\alpha_G k_B T \delta_{\alpha\beta} \tag{32}$$

and what really happens is that, in fact, the cutoff-dependent positive term is usually larger than the asymptotic negative term, except at zero temperature when the two effects cancel each other.

This behavior is most easily understood if we notice that the integral in the definition of φ, Eq. (26), can be also written as

$$\varphi(t) = k_B T \frac{d}{dt} \coth(\pi k_B T t), \tag{33}$$

and take into account, as pointed out by Ford and O'Connel,[24,25] that the correct formula for the derivative of the hyperbolic cotangent is, in fact,

$$\frac{d \coth x}{dx} = -\text{csch}^2 x + 2\delta(x). \tag{34}$$

The results are identical in both approaches, but the second is more economical.

The existence of a thermal correlation time, which in standard units is $\tau_{\text{th}} = h/k_B T = (1.27 \times 10^{-12}\ \text{s})/T$, implies that the approximation of a Markovian description for the random field is only valid for time scales which are longer than τ_{th} or in the limit of high temperatures. Thus, in the problem of magnetization control using ultrashort laser pulses it might be not a good approximation to consider the process as Markovian. Equation (32) is remarkable because it is a manifestation of a fluctuation-dissipation theorem.

We also note that, with our assumptions regarding the constants $H_\alpha(\omega)$ [that is, that $H_{\alpha\mu}(\mathbf{k}) = H_\alpha(\omega_\mu(\mathbf{k}))$ and Eq. (24) valid in all temperature regimes], we can always recover the Landau-Lifshitz-Gilbert form of the equations of motion in the limit of $\mathbf{D} \to 0$. To see it we just need to note that, using Eq. (A30), with $\mathbf{D} \to 0$,

$$\left.\frac{\delta S_{\text{eff}}}{\delta \mathbf{D}(s)}\right|_{\mathbf{D}=0} = \int_{t_0}^{t} dt' \mathbf{K}(s - t') \mathbf{S}(t'), \tag{35}$$

where the matrix $\mathbf{K}$ arises from the response components of the closed time path Green's function [see Eq. (A27)] and has matrix elements

$$K_{\alpha\beta}(t) = 2i \sum_{\mathbf{k},\mu} \text{Re}\{H^*_{\alpha\mu}(\mathbf{k}) H_{\beta\mu}(\mathbf{k})\}$$
$$\times e^{-i\omega_\mu(\mathbf{k})t} \theta(-t). \tag{36}$$

If we use all these assumptions, the last expression becomes

$$G_{\alpha\beta}(t) = 2\alpha_G \delta_{\alpha\beta} \theta(-t) \frac{d}{dt} \delta(t), \tag{37}$$

which yields the result

$$\left.\frac{\delta S_{\text{eff}}}{\delta \mathbf{D}(s)}\right|_{\mathbf{D}(s)=0} = \alpha_G \dot{\mathbf{S}}(s), \tag{38}$$

i.e., the Gilbert damping term, so that the equation of motion for the spin becomes always the Landau-Lifshitz-Gilbert equation in the limit in which the random field is neglected. In the case of high temperatures we also recover the Landau-Lifshitz-Gilbert-Brown equation with the random field and in view of the results of Garanin[14] we are able to recover the Landau-Lifshitz-Bloch equation for the average over spin fluctuations. For this purpose we can neglect the term in the equation of motion which couples two spin fields and a random field.[14] This is valid in the limit of small fluctuations.

III. SPIN-ELECTRON THEORY

In this model the system can be viewed as a set of spins on a lattice, each spin written in the form of a general spin-j representation of the SU(2) group, modeling $4d$-type electrons in a magnetic medium. We consider the reservoir to be composed of conduction electrons. The interaction Hamiltonian is an s-d-type interaction of the conduction electrons with the spin vector field.

The model Hamiltonian we consider here has the form of Eq. (1), where

$$\hat{\mathcal{H}}_S = -\sum_i \mathbf{H}_i \cdot \hat{\mathbf{S}}_i - \frac{1}{2} \sum_{ij} J_{ij} \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j, \quad J_{ij} \geqslant 0, \tag{39}$$

or, in the momentum representation,

$$\hat{\mathcal{H}}_S = -\sum_{\mathbf{k}} \mathbf{H}(-\mathbf{k}) \cdot \hat{\mathbf{S}}(\mathbf{k}) - \frac{1}{2} \sum_{\mathbf{k}} J(\mathbf{k}) \hat{\mathbf{S}}(-\mathbf{k}) \cdot \hat{\mathbf{S}}(\mathbf{k}). \tag{40}$$

The other terms are given by

$$\hat{\mathcal{H}}_R = \sum_{\mathbf{k},\alpha} \epsilon(\mathbf{k}) \hat{c}^\dagger_\alpha(\mathbf{k}) \hat{c}_\alpha(\mathbf{k}) - \lambda \sum_{\mathbf{k}} \hat{\mathbf{s}}(-\mathbf{k}) \cdot \hat{\mathbf{S}}(\mathbf{k})$$
$$- \sum_{\mathbf{k}} \hat{\mathbf{s}}(-\mathbf{k}) \cdot \mathbf{h}(\mathbf{k}), \tag{41}$$

where $\epsilon(\mathbf{k})$ is the energy of the conduction electrons, $\mathbf{h}(\mathbf{k})$ denotes the magnetic field felt by the electrons, and $\hat{\mathbf{s}}(\mathbf{k})$

denotes the composite operator

$$\hat{\mathbf{s}}(\mathbf{k}) = \frac{1}{2}\sum_{\mathbf{k}',\alpha,\beta} \hat{c}^{\dagger}_{\alpha}(\mathbf{k}'-\mathbf{k})(\boldsymbol{\sigma})_{\alpha\beta}\hat{c}_{\beta}(\mathbf{k}'), \tag{42}$$

which is the momentum representation of the spin density operator of conduction electrons. In the above equation, $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)^T$ is the vector whose components are the Pauli matrices. The Latin indices, $\{i,j,k,\ldots\}$, refer to space indices and the Greek indices, $\{\alpha,\beta,\ldots\}$, refer to spin indices. Furthermore, the operators $\hat{c}_{\alpha}(\mathbf{k})$, $\hat{c}^{\dagger}_{\alpha}(\mathbf{k})$ are electron annihilation and creation operators, which satisfy the algebra

$$\{\hat{c}_{\alpha}(\mathbf{k}_1),\hat{c}^{\dagger}_{\beta}(\mathbf{k}_2)\} = \delta_{\alpha\beta}(2\pi)^3\delta^3(\mathbf{k}_1-\mathbf{k}_2), \tag{43}$$

$$\{\hat{c}_{\alpha}(\mathbf{k}_1),\hat{c}_{\beta}(\mathbf{k}_2)\} = \{\hat{c}^{\dagger}_{\alpha}(\mathbf{k}_1),\hat{c}^{\dagger}_{\beta}(\mathbf{k}_2)\} = 0. \tag{44}$$

We now consider the path-integral representation of the reduced density matrix associated with this system, as we did in Sec. II. As so, we are going to use the basis of coherent states for the system and the reservoir (in this case we have to use Grassmann variables for fermions), so that the states of the Hilbert space are written in the form

$$|\mathbf{S}\rangle \otimes ||\gamma\rangle, \tag{45}$$

where

$$|\mathbf{S}\rangle = \prod_{\mathbf{k}} |\mathbf{S}(\mathbf{k})\rangle = \prod_{\mathbf{k}} \hat{\mathcal{D}}(\mathbf{S}(\mathbf{k}))|0\rangle, \tag{46}$$

$$||\gamma\rangle = \prod_{\alpha,\mathbf{k}} ||\gamma_{\alpha}(\mathbf{k})\rangle = \prod_{\alpha,\mathbf{k}} \exp(\hat{c}^{\dagger}_{\alpha}(\mathbf{k})\gamma_{\alpha}(\mathbf{k}))|0\rangle. \tag{47}$$

The states $|\mathbf{S}\rangle$ are to be defined in such a way that they satisfy $\langle\mathbf{S}|\hat{\mathbf{S}}(\mathbf{k})|\mathbf{S}\rangle = \mathbf{S}(\mathbf{k})\langle\mathbf{S}|\hat{\mathbb{1}}_S|\mathbf{S}\rangle$. Let us now consider the state

$$|\mathbf{S}'\rangle = \prod_i (1+|\zeta(\mathbf{S}_i)|^2)^{-j}\exp(\zeta(\mathbf{S}_i)\hat{S}_{-,i})|\psi_0\rangle, \tag{48}$$

where $|\psi_0\rangle$ is the tensor product of highest weight states of the spin-j representation and $\zeta(\mathbf{S}_i) = \tan(\theta_i/2)e^{i\varphi_i}$ denotes the stereographic projection of $\mathbf{S}_i = j\,(\sin\theta_i\cos\varphi_i, \sin\theta_i\sin\varphi_i, \cos\theta_i)^T$ through the north pole. Clearly, this state satisfies

$$\langle\mathbf{S}'|\hat{\mathbf{S}}_i|\mathbf{S}'\rangle = \mathbf{S}_i\langle\mathbf{S}'|\hat{\mathbb{1}}_S|\mathbf{S}'\rangle = \mathbf{S}_i, \tag{49}$$

and we can now replace in the last equation $\hat{\mathbf{S}}_i$ and $\mathbf{S}_i$ by their Fourier representations, yielding

$$\langle\mathbf{S}'|\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}_i}\hat{\mathbf{S}}(\mathbf{k})|\mathbf{S}'\rangle = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}_i}\mathbf{S}(\mathbf{k}) \tag{50}$$

or

$$\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}_i}\langle\mathbf{S}'|(\hat{\mathbf{S}}(\mathbf{k})-\mathbf{S}(\mathbf{k}))|\mathbf{S}'\rangle = 0. \tag{51}$$

Since this relation is valid for all $\mathbf{x}_i$, we identify $|\mathbf{S}'\rangle$ with $|\mathbf{S}\rangle$ because

$$\langle\mathbf{S}'|\hat{\mathbf{S}}(\mathbf{k})|\mathbf{S}'\rangle = \mathbf{S}(\mathbf{k})\langle\mathbf{S}'|\hat{\mathbb{1}}_S|\mathbf{S}'\rangle. \tag{52}$$

Assuming that at time t_0 the spins and electron systems are decoupled and the reduced density matrix of the bath, at that time, takes the form of a Boltzmann factor with Hamiltonian $\hat{\mathcal{H}}_R$ and inverse temperature β, as given by Eq. (A2), we easily find, in analogy with Sec. II, replacing $\alpha \to \gamma$ and taking special care because the γ's are Grassmann variables, that the expression for the propagator is the same as Eq. (A13) but now in $I[\mathbf{S}_1,\mathbf{S}_2]$ one must sum over all spin-j degrees of freedom and the Feynman-Vernon functional is given by

$$\mathcal{F}[\mathbf{S}_1,\mathbf{S}_2] = \int d\mu(\gamma_0)d\mu(\gamma_1)d\mu(\gamma_2)\, k(-\gamma_0^{\dagger},t;\gamma_1,t_0|\mathbf{S}_1) \times Z_R^{-1}k(\gamma_1^{\dagger},-i\beta;\gamma_2,0|0) \times k^*(\gamma_0^{\dagger},t;\gamma_2,t_0|\mathbf{S}_2), \tag{53}$$

where $d\mu(\gamma) = \prod_{\alpha,\mathbf{k}} d^2\gamma_{\alpha}(\mathbf{k})\exp(-\gamma^*_{\alpha}(\mathbf{k})\gamma_{\alpha}(\mathbf{k}))$ and the kernel $k(\gamma_f^{\dagger},t_f;\gamma_i,t_i|\mathbf{S})$ is defined in the same way as in Eq. (A19) (again, taking special care because Grassmann variables anticommute). The minus sign in the first kernel in Eq. (53) is due to the antiperiodic conditions on the trace formula using fermionic coherent states.

Unlike in the case of the phonons, we cannot compute the Gaussian integrals exactly. We would like then to do some expansion depending on the parameter λ. In order to achieve that we perform a Hubbard-Stratonovich transformation and then expand to second order a determinant resulting from the functional integrals. To do so, one can define an auxiliary bilinear form

$$(G^{-1})_{\alpha\beta}(t-t',\mathbf{k}-\mathbf{k}';\mathbf{S}) = -\tfrac{1}{2}\delta(t-t')[\boldsymbol{\sigma}_{\alpha\beta}\cdot\mathbf{h}(\mathbf{k}-\mathbf{k}') + \lambda\boldsymbol{\sigma}_{\alpha\beta}\cdot\mathbf{S}(t,\mathbf{k}-\mathbf{k}')]. \tag{54}$$

With this definition, it is clear that we can do the Hubbard-Stratonovich transformation of the form

$$\begin{aligned}
&\exp\left(-i\int_{t_0}^{t}\int_{t_0}^{t} ds ds' \sum_{\mathbf{k}\mathbf{k}',\alpha\beta}\gamma^*_{\alpha}(s,\mathbf{k})(G^{-1})_{\alpha\beta} \times (s-s',\mathbf{k}-\mathbf{k}';\mathbf{S})\gamma_{\beta}(s',\mathbf{k}')\right)\\
&= \int \frac{\prod_{\alpha,\mathbf{k}}\mathcal{D}^2\zeta_{\alpha}(\mathbf{k})}{\det(-iG)[\mathbf{S}]}\exp\left(i\int_{t_0}^{t}\int_{t_0}^{t} ds ds' \times \sum_{\mathbf{k}\mathbf{k}',\alpha\beta}\zeta^*_{\alpha}(s,\mathbf{k})(G)_{\alpha\beta}(s-s',\mathbf{k}-\mathbf{k}';\mathbf{S})\zeta_{\beta}(s',\mathbf{k}') + i\int_{t_0}^{t}\int_{t_0}^{t} ds\sum_{\mathbf{k},\alpha}\zeta^*_{\alpha}(s,\mathbf{k})\gamma_{\alpha}(s,\mathbf{k}) + \gamma^*_{\alpha}(s,\mathbf{k})\zeta_{\alpha}(s,\mathbf{k})\right).
\end{aligned} \tag{55}$$

By doing this, we achieve a linear coupling between the γ's and the ζ's, which allows us to compute the Gaussian integrals in the γ's. Replacing this transformation in the expression for the influence functional and computing the Gaussian integrals associated with the γ's, we arrive at

$$\int \frac{\prod_{\alpha,\mathbf{k}}\mathcal{D}^2\zeta_{1,\alpha}(\mathbf{k})}{\det(-iG)[\mathbf{S}_1]}\frac{\prod_{\alpha,\mathbf{k}}\mathcal{D}^2\zeta_{2,\alpha}(\mathbf{k})}{\det(iG)[\mathbf{S}_2]} \times \exp\left(i\int_{t_0}^{t}\int_{t_0}^{t} ds ds' \sum_{\mathbf{k}\mathbf{k}',\alpha\beta}(\zeta^*_{1,\alpha}(s,\mathbf{k})\;\zeta^*_{2,\alpha}(s,\mathbf{k})) \times (\Delta(s-s',\mathbf{k}-\mathbf{k}')_{\alpha\beta})\begin{pmatrix}\zeta_{1,\beta}(s',\mathbf{k}')\\ \zeta_{2,\beta}(s',\mathbf{k}')\end{pmatrix}\right), \tag{56}$$

where

$$(\Delta(t,\mathbf{k})_{\alpha\beta}) = \begin{pmatrix} (\mathcal{G}_{11})_{\alpha\beta}(t,\mathbf{k}) + (G)_{\alpha\beta}(t,\mathbf{k};\mathbf{S}_1) & (\mathcal{G}_{12})_{\alpha\beta}(t,\mathbf{k}) \\ (\mathcal{G}_{21})_{\alpha\beta}(t,\mathbf{k}) & (\mathcal{G}_{22})_{\alpha\beta}(t,\mathbf{k}) - (G)_{\alpha\beta}(t,\mathbf{k};\mathbf{S}_2) \end{pmatrix}, \tag{57}$$

in which

$$\begin{pmatrix} (\mathcal{G}_{11})_{\alpha\beta}(t,\mathbf{k}) & (\mathcal{G}_{12})_{\alpha\beta}(t,\mathbf{k}) \\ (\mathcal{G}_{21})_{\alpha\beta}(t,\mathbf{k}) & (\mathcal{G}_{22})_{\alpha\beta}(t,\mathbf{k}) \end{pmatrix}
= i(2\pi)^3\delta^3(\mathbf{k})\delta_{\alpha\beta}e^{-i\epsilon(\mathbf{k})t} \begin{pmatrix} (1-f(\epsilon(\mathbf{k})))\theta(t) - f(\epsilon(\mathbf{k}))\theta(-t) & -f(\epsilon(\mathbf{k})) \\ 1-f(\epsilon(\mathbf{k})) & (1-f(\epsilon(\mathbf{k})))\theta(-t) - f(\epsilon(\mathbf{k}))\theta(t) \end{pmatrix}, \tag{58}$$

with $f(x) = (e^x+1)^{-1}$ being the Fermi-Dirac distribution.

The functional integral is readily evaluated to be

$$\det\left\{\left[-i\begin{pmatrix} \mathcal{G}_{11}+G(\mathbf{S}_1) & \mathcal{G}_{12} \\ \mathcal{G}_{21} & \mathcal{G}_{22}-G(\mathbf{S}_2)\end{pmatrix}\right] \times \left[i\begin{pmatrix} G^{-1}(\mathbf{S}_1) & 0 \\ 0 & -G^{-1}(\mathbf{S}_2)\end{pmatrix}\right]\right\} = \det(\mathbb{1}+\mathfrak{S}), \tag{59}$$

where

$$\mathfrak{S} = \begin{pmatrix} \mathcal{G}_{11}G^{-1}(\mathbf{S}_1) & -\mathcal{G}_{12}G^{-1}(\mathbf{S}_2) \\ \mathcal{G}_{21}G^{-1}(\mathbf{S}_1) & -\mathcal{G}_{22}G^{-1}(\mathbf{S}_2)\end{pmatrix}. \tag{60}$$

The determinant of Eq. (59) should be understood, of course, in the functional sense. We can write

$$\det(\mathbb{1}+\mathfrak{S}) = \exp\{\mathrm{Tr}[\log(\mathbb{1}+\mathfrak{S})]\} = \exp\left(\sum_k \frac{(-1)^{k+1}}{k}\mathrm{Tr}\mathfrak{S}^k\right), \tag{61}$$

where the trace is also understood in the functional sense. In the way it is written, this produces an expansion in powers of the matrix elements of $\mathfrak{S}$ and consequently an expansion in the parameter λ.

The first term of the expansion is easily found to be zero and if we keep only terms of second order in the matrix elements of $\mathfrak{S}$ we obtain, in the Keldysh representation of the fields as defined by Eq. (A25),

$$\begin{aligned}
\log[\det(\mathbb{1}+\mathfrak{S})] &= -\frac{1}{2}\mathrm{Tr}\mathfrak{S}^2 + O(\mathfrak{S}^3) \\
&= -\lambda\sum_{\mathbf{k}}\int_{t_0}^t\int_{t_0}^t dt_1dt_2(\mathcal{S}g)(t_1-t_2,\mathbf{k})\theta(t_2-t_1) \\
&\quad \times \mathbf{h}(-\mathbf{k})\cdot\mathbf{D}(t_2,\mathbf{k}) \\
&\quad - \lambda^2\sum_{\mathbf{k}}\int_{t_0}^t\int_{t_0}^t dt_1dt_2(\mathcal{S}g)(t_1-t_2,\mathbf{k})\theta(t_2-t_1) \\
&\quad \times \mathbf{S}(t_1,-\mathbf{k})\cdot\mathbf{D}(t_2,\mathbf{k}) \\
&\quad - \frac{\lambda^2}{4}\sum_{\mathbf{k}_1}\int_{t_0}^t\int_{t_0}^t dt_1dt_2(\mathcal{P}g)(t_1-t_2,\mathbf{k})\mathbf{D}(t_1,-\mathbf{k})\cdot \\
&\quad \times \mathbf{D}(t_2,\mathbf{k}) + O(\mathfrak{S}^3),
\end{aligned} \tag{62}$$

where

$$g(t,\mathbf{k}) = \sum_{\mathbf{k}'} e^{-i(\epsilon(\mathbf{k}')-\epsilon(\mathbf{k}'-\mathbf{k}))t} \times (1-f(\epsilon(\mathbf{k}')))f(\epsilon(\mathbf{k}'-\mathbf{k})), \tag{63}$$

and

$$\begin{aligned}(\mathcal{P}\phi)(t,\mathbf{k}) &= \tfrac{1}{2}(\phi(t,\mathbf{k})+\phi(-t,-\mathbf{k})), \\ (\mathcal{S}\phi)(\mathbf{k}) &= \tfrac{1}{2}(\phi(t,\mathbf{k})-\phi(-t,-\mathbf{k})),\end{aligned} \tag{64}$$

are the symmetrizer and the antisymmetrizer operators associated with the time and momentum variables, respectively. This makes it clear that the part leading to dissipation will be the term coupling two **D** fields since it will introduce an imaginary term in the action. This is easy to see because the Fourier transform of an even (odd) function is real (imaginary) if the function is real.

The effective action coming from the electron contribution thus obtained can be compactly written as

$$-\frac{1}{2}\sum_{\mathbf{k}}\int_{t_0}^t dt_1\int_{t_0}^t dt_2\left(\sqrt{2}S_\alpha(t_1,-\mathbf{k})\ \frac{1}{\sqrt{2}}D_\alpha(t_1,-\mathbf{k})\right) \times \sigma_1\tilde{G}^{\alpha\beta}(t_1-t_2,\mathbf{k})\sigma_1\begin{pmatrix}\sqrt{2}S_\beta(t_2,\mathbf{k}) \\ \frac{1}{\sqrt{2}}D_\beta(t_2,\mathbf{k})\end{pmatrix}, \tag{65}$$

where

$$\begin{aligned} i\tilde{G}_{\alpha\beta}(t,\mathbf{k}) &= \begin{pmatrix} (\mathcal{P}g)(t,\mathbf{k}) & -(\mathcal{S}g)(t,\mathbf{k})\theta(t) \\ (\mathcal{S}g)(t,\mathbf{k})\theta(-t) & 0\end{pmatrix} \\ &:= i\begin{pmatrix} G^K(t,\mathbf{k}) & G^R(t,\mathbf{k}) \\ G^A(t,\mathbf{k}) & 0\end{pmatrix}.\end{aligned} \tag{66}$$

We now prove, for consistency, that the following fluctuation dissipation theorem holds:

$$G^K(\omega,\mathbf{k}) = (G^R(\omega,\mathbf{k}) - G^A(\omega,\mathbf{k}))\coth\left(\frac{\beta\omega}{2}\right). \tag{67}$$

Computing the time Fourier transform of $(\mathcal{P}g)(t,\mathbf{k})$, we end up with

$$\begin{aligned}(\mathcal{P}g)(\omega,\mathbf{k}) &= \frac{1}{2}\sum_{\mathbf{k}'}2\pi\delta(\omega-\epsilon(\mathbf{k}-\mathbf{k}')+\epsilon(\mathbf{k}')) \\ &\quad\times[(1-f(\epsilon(\mathbf{k}'))f(\epsilon(\mathbf{k}-\mathbf{k}')) \\ &\quad -(1-f(\epsilon(\mathbf{k}-\mathbf{k}')))f(\epsilon(\mathbf{k}'))].\end{aligned} \tag{68}$$

We now recall the identity

$$f(\epsilon)(1-f(\epsilon-\omega)) = n(\omega)(f(\epsilon-\omega)-f(\epsilon)), \quad (69)$$

where $n(\omega)$ is the Bose-Einstein distribution, and we observe that we can write

$$\begin{aligned}(1-f(\epsilon-\omega))f(\epsilon)+(1-f(\epsilon))f(\epsilon-\omega) &= (1-f(\epsilon-\omega))f(\epsilon)+f(\epsilon-\omega)\\ &\quad -f(\epsilon)f(\epsilon-\omega)+f(\epsilon)-f(\epsilon)\\ &= 2(1-f(\epsilon-\omega))f(\epsilon)+f(\epsilon-\omega)-f(\epsilon)\\ &= 2n(\omega)(f(\epsilon-\omega)-f(\epsilon))+(f(\epsilon-\omega)-f(\epsilon))\\ &= (2n(\omega)+1)(f(\epsilon-\omega)-f(\epsilon))\\ &= \coth\left(\frac{\beta\omega}{2}\right)(f(\epsilon-\omega)-f(\epsilon)). \quad (70)\end{aligned}$$

Now using $f(\omega)\delta(\omega-\omega_0) = f(\omega_0)\delta(\omega-\omega_0)$, we find

$$(\mathcal{P}g)(\omega,\mathbf{k}) = \coth\left(\frac{\beta\omega}{2}\right)\frac{1}{2}\sum_{\mathbf{k}'}2\pi\delta(\omega-\epsilon(\mathbf{k}-\mathbf{k}')+\epsilon(\mathbf{k}')) \times[f(\epsilon(\mathbf{k}'))-f(\epsilon(\mathbf{k}-\mathbf{k}'))]. \quad (71)$$

Since

$$G^R(t,\mathbf{k})-G^A(t,\mathbf{k}) = i(\mathcal{S}g)(t,\mathbf{k}), \quad (72)$$

we compute the time Fourier transform of $(\mathcal{S}g)(t,\mathbf{k})$, obtaining

$$\begin{aligned}(\mathcal{S}g)(\omega,\mathbf{k}) &= \frac{1}{2}\sum_{\mathbf{k}'}2\pi\delta(\omega-\epsilon(\mathbf{k}-\mathbf{k}')-\epsilon(\mathbf{k}'))\\ &\quad\times[f(\epsilon(\mathbf{k}-\mathbf{k}'))(1-f(\epsilon(\mathbf{k}')))\\ &\quad-(1-f(\epsilon(\mathbf{k}-\mathbf{k}')))f(\epsilon(\mathbf{k}'))]\\ &= \frac{1}{2}\sum_{\mathbf{k}'}2\pi\delta(\omega-\epsilon(\mathbf{k}-\mathbf{k}')-\epsilon(\mathbf{k}'))\\ &\quad\times[f(\epsilon(\mathbf{k}-\mathbf{k}'))-f(\epsilon(\mathbf{k}'))]\\ &= -\frac{1}{2}\sum_{\mathbf{k}'}2\pi\delta(\omega-\epsilon(\mathbf{k}-\mathbf{k}')-\epsilon(\mathbf{k}'))\\ &\quad\times[f(\epsilon(\mathbf{k}'))-f(\epsilon(\mathbf{k}-\mathbf{k}'))], \quad (73)\end{aligned}$$

which implies

$$(\mathcal{P}g)(\omega,\mathbf{k}) = -\coth\left(\frac{\beta\omega}{2}\right)(\mathcal{S}g)(\omega,\mathbf{k}), \quad (74)$$

so that Eq. (67) holds as we originally claimed.

One can, analogously to what we did in Sec. II, introduce a random field using a Hubbard-Stratonovich transformation. This time the correlation function in position space reads

$$\begin{aligned}\langle\xi_\alpha(t,\mathbf{x})\xi_\beta(0)\rangle &= \delta_{\alpha\beta}\frac{1}{2}[(\mathcal{F}^{-1}\circ G^K](t,\mathbf{x})\\ &= \delta_{\alpha\beta}\frac{\lambda^2}{2}[(\mathcal{F}^{-1}\circ\mathcal{P})g](t,\mathbf{x})\\ &= \delta_{\alpha\beta}\frac{\lambda^2}{2}\mathrm{Re}\{[\mathcal{F}^{-1}g](t,\mathbf{x})\}, \quad (75)\end{aligned}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform associated with the variable $\mathbf{k}$.

There is another fluctuation dissipation theorem which holds for this particular effective theory which we prove in the following. In the case of Brownian motion, the time integral of the noise correlation function is proportional to the temperature with the constant of proportionality being the damping constant. We prove that also here, in the case of a spin system interacting with a bath of conduction electrons, a relation of the same type holds. In order to prove this, we integrate the correlation function of Eq. (75),

$$\int_{-\infty}^{\infty}ds\int d^3x\langle\xi_\alpha(s,\mathbf{x})\xi_\beta(0)\rangle = \delta_{\alpha\beta}\frac{\lambda^2}{2}\mathrm{Re}\left\{\int_{-\infty}^{\infty}ds\int d^3x[\mathcal{F}^{-1}g](s,\mathbf{x})\right\}. \quad (76)$$

Now we rewrite the integral appearing in the above expression by replacing the expression for $g(t,\mathbf{k})$ of Eq. (63),

$$\int_{-\infty}^{\infty}ds\int d^3x[\mathcal{F}^{-1}g](s,\mathbf{x}) = \int_{t_0}^{t}ds\int d^3x\sum_{\mathbf{k}\mathbf{k}'}e^{i[\mathbf{k}\cdot\mathbf{x}-i(\epsilon(\mathbf{k}')-\epsilon(\mathbf{k}'-\mathbf{k}))s]} \times(1-f(\epsilon(\mathbf{k}')))f(\epsilon(\mathbf{k}'-\mathbf{k})). \quad (77)$$

Identifying the Dirac δ's appearing in the above expression, we find

$$\sum_{\mathbf{k}\mathbf{k}'}(2\pi)^3\delta^3(\mathbf{k})\delta(\epsilon(\mathbf{k}')-\epsilon(\mathbf{k}'-\mathbf{k})) \times(1-f(\epsilon(\mathbf{k}')))f(\epsilon(\mathbf{k}'-\mathbf{k})), \quad (78)$$

or, if we define the excitation energy $\omega_{\mathbf{k}',\mathbf{k}} = \epsilon(\mathbf{k}'-\mathbf{k})-\epsilon(\mathbf{k}')$, we get

$$\sum_{\mathbf{k}\mathbf{k}'}(2\pi)^3\delta^3(\mathbf{k})\delta(\omega_{\mathbf{k}',\mathbf{k}}) \times(1-f(\epsilon(\mathbf{k}')))f(\epsilon(\mathbf{k}')+\omega_{\mathbf{k}',\mathbf{k}}). \quad (79)$$

The above formula has to be treated with some care. Recalling the identity of Eq. (69), we obtain

$$\sum_{\mathbf{k}\mathbf{k}'}(2\pi)^3\delta^3(\mathbf{k})\delta(\omega_{\mathbf{k}',\mathbf{k}}) \times n(\omega_{\mathbf{k}',\mathbf{k}})(f(\epsilon(\mathbf{k}'))-f(\epsilon(\mathbf{k}')+\omega_{\mathbf{k}',\mathbf{k}})). \quad (80)$$

Thanks to the δ function associated with the excitation energy, we only need the integrand evaluated at $\omega_{\mathbf{k}',\mathbf{k}} = 0$. Clearly, the expression is ill determined because the Bose-Einstein distribution diverges and the difference of Fermi-Dirac distributions goes to zero. To proceed we have to expand the integrand for small $\omega_{\mathbf{k}',\mathbf{k}}$. Observing that $f'(\epsilon) = -f(\epsilon)(1-f(\epsilon))$ we can Taylor expand the part containing Fermi-Dirac distributions so that we obtain

$$\begin{aligned}n(\omega_{\mathbf{k}',\mathbf{k}})(f(\epsilon(\mathbf{k}'))-f(\epsilon(\mathbf{k}')+\omega_{\mathbf{k}',\mathbf{k}})) &= \frac{1}{\beta\omega_{\mathbf{k}',\mathbf{k}}}[\omega_{\mathbf{k}',\mathbf{k}}f(\epsilon(\mathbf{k}'))(1-f(\epsilon(\mathbf{k}')))]\\ &\quad+O(\omega_{\mathbf{k}',\mathbf{k}})\\ &= \frac{1}{\beta}f(\epsilon(\mathbf{k}'))(1-f(\epsilon(\mathbf{k}')))+O(\omega_{\mathbf{k}',\mathbf{k}}). \quad (81)\end{aligned}$$

This means that our sum becomes simply

$$\frac{1}{\beta}\sum_{\mathbf{k}\mathbf{k}'}(2\pi)^3\delta^3(\mathbf{k})\delta(\omega_{\mathbf{k}',\mathbf{k}}) \times f(\epsilon(\mathbf{k}'))(1-f(\epsilon(\mathbf{k}'))). \tag{82}$$

If we sum over $\mathbf{k}'$ the only contribution of the sum will be the one in which $\omega_{\mathbf{k}',\mathbf{k}}=0$, or $\epsilon(\mathbf{k}'-\mathbf{k})=\epsilon(\mathbf{k})$. Assuming that $\epsilon(\mathbf{k})$ is a power-law function of $|\mathbf{k}|$, this implies that the only term surviving is the one with $\mathbf{k}'=0$. This reasoning gives

$$\frac{1}{\beta}f(\epsilon(0))(1-f(\epsilon(0)))\sum_{\mathbf{k}}(2\pi)^3\delta^3(\mathbf{k}) = \frac{1}{\beta}f(\epsilon(0))(1-f(\epsilon(0))) = \frac{1}{4\beta}. \tag{83}$$

Thus, the final result is

$$\int_{-\infty}^{\infty}ds\int d^3x\langle\xi_\alpha(s,\mathbf{x})\xi_\beta(0)\rangle = \int_{-\infty}^{\infty}ds\int d^3x\,\mathrm{Re}\{\langle\xi_\alpha(s,\mathbf{x})\xi_\beta(0)\rangle\} = 2\alpha'_G k_B T\delta_{\alpha\beta}, \tag{84}$$

where we have defined $\alpha'_G=\lambda^2/16$. The above formula shows that a fluctuation-dissipation theorem holds. Although a fluctuation-dissipation theorem is satisfied in the exact treatment,[26,27] this is an important verification since in our treatment we were forced to make the lowest-order nonperturbative approximation which is equivalent to a linear response theory. In the next two sections, we explore the relevance of the result obtained in Eq. (75) for the electron contribution to the noise correlation function by showing its relation with the Lindblad equation obtained under the Born-Markov approximation, by computing its exact analytical form and studying it numerically.

A. Relation to the Lindblad equation

In the Born-Markov approximation one can derive the so-called Lindblad equation for the reduced density matrix of the system in the interaction picture, $\hat{\rho}_{\mathrm{S}}(t)$,[20]

$$\frac{\partial\hat{\rho}_{\mathrm{S}}}{\partial t} = -i\left[\sum_{\omega,\mathbf{k},\mathbf{k}',\alpha\beta}\Omega_{\alpha\beta}(\omega,\mathbf{k},\mathbf{k}')(\hat{S}^\alpha(\omega,\mathbf{k}))^\dagger\hat{S}^\beta(\omega,\mathbf{k}'),\hat{\rho}_{\mathrm{S}}\right] + \sum_{\omega,\mathbf{k},\mathbf{k}',\alpha,\beta}\Gamma_{\alpha\beta}(\omega,\mathbf{k},\mathbf{k}')\big[\hat{S}^\beta(\omega,\mathbf{k}')\hat{\rho}_{\mathrm{S}}(\hat{S}^\alpha(\omega,\mathbf{k}))^\dagger - \frac{1}{2}\{(\hat{S}^\alpha(\omega,\mathbf{k}))^\dagger\hat{S}^\beta(\omega,\mathbf{k}'),\hat{\rho}_{\mathrm{S}}\}\big], \tag{85}$$

in which

$$\hat{S}^\alpha(\omega,\mathbf{k}) := \sum_{E-E'=\omega}P_{E'}\hat{S}^\alpha(\mathbf{k})P_E, \tag{86}$$

where $P_E=|E\rangle\langle E|$ are projectors onto energy eigenstates ($\hat{\mathcal{H}}_{\mathrm{S}}|E\rangle=E|E\rangle$) of the system Hamiltonian. The tensors $i\Omega_{\alpha\beta}$ and $\Gamma_{\alpha\beta}$ are, respectively, the anti-Hermitian and twice the Hermitian part of the tensor

$$J_{\alpha\beta}(\omega,\mathbf{k},\mathbf{k}') := \int_0^\infty ds\; C_{\alpha\beta}(s,\mathbf{k},\mathbf{k}')e^{i\omega s}, \tag{87}$$

$$C_{\alpha\beta}(s,\mathbf{k},\mathbf{k}') := \lambda^2\mathrm{Tr}\{\hat{\rho}_{\mathrm{R}}(\hat{s}_\alpha(\mathbf{k},t))^\dagger\hat{s}_\beta(\mathbf{k}',t-s)\}, \tag{88}$$

in which $\hat{s}_\alpha(t,\mathbf{k})$ are the compound operators associated to the spin of the conduction-band electrons in the interaction representation. A simple application of Wick's theorem at finite temperature to Eq. (88) shows that $C_{\alpha\beta}(t,\mathbf{k},\mathbf{k}')$ is diagonal in momentum and spin variables, and the diagonal components are precisely the Fourier transform of the electron correlator, i.e.,

$$C_{\alpha\beta}(t,\mathbf{k},\mathbf{k}') = \frac{\lambda^2}{2}\delta_{\alpha\beta}(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')g(t,\mathbf{k}), \tag{89}$$

in which $g(t,\mathbf{k})$ is the same quantity defined in Eq. (63) which naturally appeared while expanding the effective action coming from the conduction electrons to second order in the self-energy. In the Markovian approximation, the two-point correlation function of the bath fields which appears in the interaction Hamiltonian has all the relevant physical information to describe the dynamics of the system density matrix. Namely, the tensor $\Omega_{\alpha\beta}$ is responsible for the so-called Lamb shift which appears in the energies of the unperturbed system[20] and the tensor $\Gamma_{\alpha\beta}$ is related to the rates in which dissipative processes occur. From Eq. (89) we find that the correlation function of the noise originating from the interaction of the conduction electrons with the spin, Eq. (75), has a very deep physical meaning: it carries all the physical information on the shifts of the system energies and the rates of the nonconservative processes due to the interaction with the bath in the Markovian approximation. By measuring it we can, thus, extract the relevant physical properties of the system and the interaction in the Markovian limit.

B. Analyzing the electron contribution for $\langle\xi_\alpha(t,\mathbf{x})\xi_\beta(0)\rangle$

In this section, we derive the exact analytical form of the electron contribution to the noise correlator obtained given in Eq. (75) and study it numerically. To make contact with the notation used in the literature for the Keldysh-Schwinger closed-time-path Green's function formalism,[23,28] we show that the electron contribution to the correlator $\langle\xi_\alpha(t,\mathbf{x})\xi_\beta(0)\rangle$ is proportional to the Keldysh component of the polarization insertion in lowest order. To see that this is the case, we start with the formula for the Keldysh component of the polarization tensor, which is given by[28]

$$\Pi^K(t,\mathbf{k}) \propto \sum_{\mathbf{k}'}[\mathcal{G}^K(t,\mathbf{k}-\mathbf{k}')\mathcal{G}^K(-t,\mathbf{k}') + \mathcal{G}^R(t,\mathbf{k}-\mathbf{k}')\mathcal{G}^A(-t,\mathbf{k}') + \mathcal{G}^A(t,\mathbf{k}-\mathbf{k}')\mathcal{G}^R(-t,\mathbf{k}')],$$

where

$$\mathcal{G}^R(t,\mathbf{k}) = -i\theta(t)e^{-i\epsilon(\mathbf{k})t},$$

$$\mathcal{G}^A(t,\mathbf{k}) = +i\theta(-t)e^{-i\epsilon(\mathbf{k})t},$$

$$\mathcal{G}^K(t,\mathbf{k}) = -i(1-2f(\epsilon(\mathbf{k})))e^{-i\epsilon(\mathbf{k})t}.$$

These last expressions yield

$$\Pi^K(t,\mathbf{k}) \propto \sum_{\mathbf{k}'} e^{-i(\epsilon(\mathbf{k}-\mathbf{k}')-\epsilon(\mathbf{k}'))t}[f(\epsilon(\mathbf{k}-\mathbf{k}'))(1-f(\epsilon(\mathbf{k}'))) + (1-f(\epsilon(\mathbf{k}-\mathbf{k}')))f(\epsilon(\mathbf{k}'))]$$

Now, since

$$\begin{aligned} g(t,-\mathbf{k}) &= \sum_{\mathbf{k}'} e^{-i[\epsilon(-\mathbf{k}-\mathbf{k}')-\epsilon(\mathbf{k}')]t}(1-f(\epsilon(-\mathbf{k}-\mathbf{k}')))f(\epsilon(\mathbf{k}')) \\ &= \sum_{\mathbf{k}'} e^{-i[\epsilon(\mathbf{k}'-\mathbf{k})-\epsilon(-\mathbf{k}')]t}(1-f(\epsilon(\mathbf{k}'-\mathbf{k})))f(\epsilon(-\mathbf{k}')) \\ &= \sum_{\mathbf{k}'} e^{-i[\epsilon(\mathbf{k}-\mathbf{k}')-\epsilon(\mathbf{k}')]t}(1-f(\epsilon(\mathbf{k}-\mathbf{k}')))f(\epsilon(\mathbf{k}')) \\ &= g(t,\mathbf{k}), \end{aligned}$$

then

$$g(-t,-\mathbf{k}) = \sum_{\mathbf{k}'} e^{i[\epsilon(\mathbf{k}-\mathbf{k}')-\epsilon(\mathbf{k}')]t}(1-f(\epsilon(\mathbf{k}-\mathbf{k}')))f(\epsilon(\mathbf{k}')), \times \sum_{\mathbf{q}} e^{i[\epsilon(\mathbf{q})-\epsilon(\mathbf{q}+\mathbf{k})]t}(1-f(\epsilon(\mathbf{q})))f(\epsilon(\mathbf{q}+\mathbf{k})),$$

$$g(-t,-\mathbf{k}) = \sum_{\mathbf{k}'} e^{-i[\epsilon(-\mathbf{k}-\mathbf{k}')-\epsilon(\mathbf{k}')]t} f(\epsilon(\mathbf{k}-\mathbf{k}'))(1-f(\epsilon(\mathbf{k}'))).$$

In the last two derivations we used the fact that the conduction electrons spectrum is even so that $\epsilon(\mathbf{k}) = \epsilon(-\mathbf{k})$. So now we can finally check that

$$\begin{aligned} \Pi^K(t,\mathbf{x}) &\propto \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}(\mathcal{P}g)(t,\mathbf{k}) \\ &= \frac{1}{2}\sum_{\mathbf{k}}[e^{i\mathbf{k}\cdot\mathbf{x}}g(t,\mathbf{k}) + e^{-i\mathbf{k}\cdot\mathbf{x}}g(-t,\mathbf{k})] \\ &= \frac{1}{2}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}g(t,\mathbf{k}) + \text{c.c.} = \text{Re}(\mathcal{F}^{-1}\circ g)(t,\mathbf{x}), \end{aligned}$$

as claimed.

One can use the free thermal Green's function to write this expression in a more tractable way. The thermal polarization insertion operator is given by

$$\mathcal{D}^0(\mathbf{x}\tau;\mathbf{x}'\tau') := \mathcal{G}^0(\mathbf{x}\tau;\mathbf{x}'\tau')\mathcal{G}^0(\mathbf{x}'\tau';\mathbf{x}\tau), \tag{90}$$

where $\mathcal{G}^0(\mathbf{x}\tau;\mathbf{x}'\tau') = \mathcal{G}^0(\mathbf{x}-\mathbf{x}',\tau-\tau')$,

$$\mathcal{G}^0(\mathbf{x},\tau) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}e^{-\epsilon(\mathbf{k})\tau}f(\epsilon(\mathbf{k})), \quad \tau < 0, \tag{91}$$

$$\mathcal{G}^0(\mathbf{x},\tau) = -\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}e^{-\epsilon(\mathbf{k})\tau} \times (1-f(\epsilon(\mathbf{k}))), \quad \tau > 0 \tag{92}$$

is the thermal free electron propagator[29] in which $f(x) = (\exp(\beta x)+1)^{-1}$ is the Fermi-Dirac distribution. If we allow $\tau = it$, the electron contribution to the correlator is found to be

$$\langle \xi_\alpha(\mathbf{x},t)\xi_\beta(0)\rangle|_{\text{el}} = -\delta_{\alpha\beta}\frac{\lambda^2}{2}\text{Re}\{\mathcal{D}^0(\mathbf{x},\tau = it)\}, \quad t > 0. \tag{93}$$

In terms of Feynman diagrams in the imaginary time formalism this is given by the graph of Fig. 1. The calculations are shown in detail in Appendix C. The results are

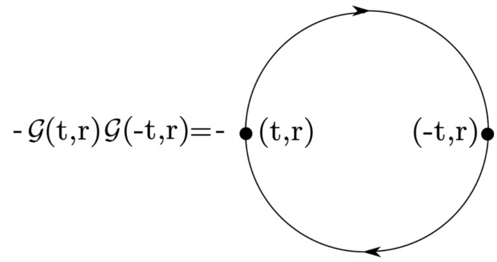

FIG. 1. Bubble diagram. The free one-particle Green's function only depends on $r = |\mathbf{x}|$, so space inversion does not affect it.

$$\mathcal{G}^0(\mathbf{x},-it) = -\frac{\mu}{\pi r\beta}\sum_{n=0}^{\infty} e^{-\sqrt{\frac{(2n+1)\mu\pi}{\beta}}r} \times \cos\left[\sqrt{\frac{(2n+1)\mu\pi}{\beta}}r + i(2n+1)\pi t\right], \tag{94}$$

and

$$\begin{aligned} -\mathcal{G}^0(|-\mathbf{x}| = |\mathbf{x}| = r, it) &= \left(\frac{\mu i}{2\pi t}\right)^{3/2}\exp\left(\frac{ir^2\mu}{2t}\right) \\ &\quad + \frac{\mu}{\pi r\beta}\sum_{n=0}^{\infty} e^{-\sqrt{\frac{(2n+1)\mu\pi}{\beta}}r} \\ &\quad \times \cos\left[\sqrt{\frac{(2n+1)\mu\pi}{\beta}}r - i(2n+1)\pi t\right], \end{aligned} \tag{95}$$

where μ is the effective mass of the electrons in the parabolic approximation. The plots in Fig. 2 give us the qualitative behavior of the electron correlator.

We can see that this noise correlation function, just by looking at $t = 0$ (when the time separation of the two noise fields is zero), is much more textured than the usual white noise correlation function as it inherits the natural Fermi-statistics structure. In particular, for small temperatures it manifests the phenomenon of electron screening because the spin acts as an impurity and, because of the sharpness of the Fermi level, only electrons near the Fermi level can scatter with the spin.

This is seen by the oscillatory behavior, Friedel oscillations, at these small temperatures. This shows that, in fact, one can use ferromagnets with impurities to probe properties of the Fermi surface of those ferromagnets just by measuring noise correlation functions in the laboratory. Later on, at higher temperatures, this phenomenon is highly suppressed. At any temperature the correlation is damped with distance, with damping constants dictated by $\sim\sqrt{\mu/\beta} \sim \sqrt{\mu T}$. At $r \to 0$ the correlation grows as $1/r^2$, for finite temperatures. This damping constant can be measured at each temperature and this allows one to estimate an effective mass for the conduction electrons.

IV. COMPOSITE MODEL

We now collect the results from Secs. II and III in a theory modeling the interaction of a spin vector field with phonons

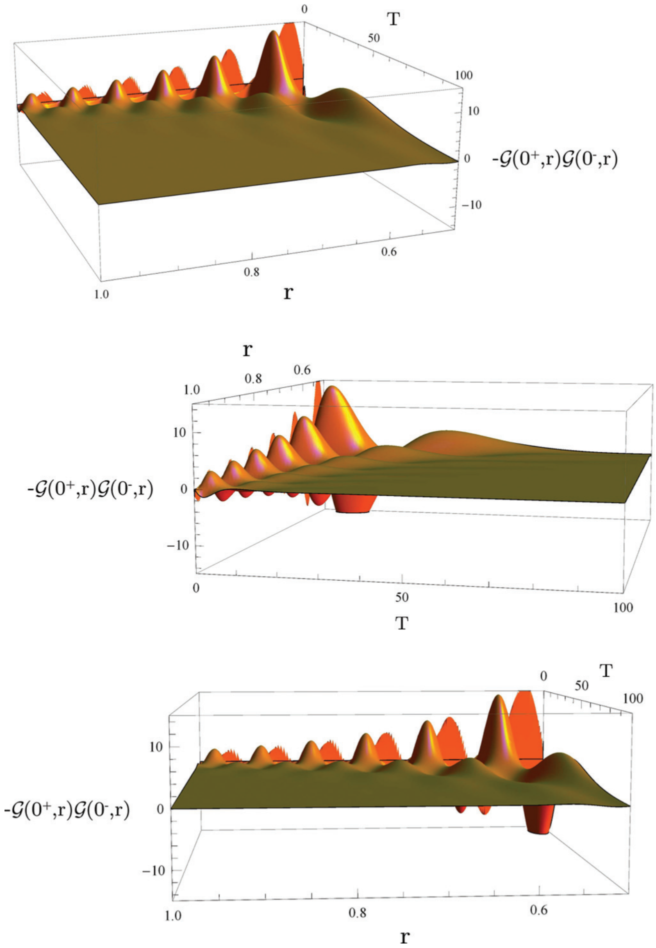

FIG. 2. (Color online) Electron contribution to the noise correlation function (minus the real part of the polarization insertion) at time $t = 0$ as a function of relative distance r and temperature of the electron bath, T. Here we have truncated the series of the Green's function in the 20th term. Natural units were used.

and electrons. It is straightforward to see that the spin-phonon interaction of Eq. (6) generalizes to

$$\hat{\mathcal{H}}_i = -\sum_{\mathbf{k}} (H^*_{\alpha\mu}(\mathbf{k})\hat{a}^\dagger_\mu(\mathbf{k})\hat{S}^\alpha(\mathbf{k}) + H_{\alpha\mu}(\mathbf{k})\hat{S}^\alpha(-\mathbf{k})\hat{a}_\mu(\mathbf{k})) \quad (96)$$

in order to account for a spin vector field (each space position now has an independent spin-j degree of freedom). The total Hamiltonian is now given by the Hamiltonian of Sec. III plus the phonon reservoir Hamiltonian and the interaction Hamiltonian.

Assuming again the factorization of the initial density matrix, the Feynman-Vernon functional of this model factorizes into the product of two functionals, one associated to the phonons and the other to the electrons:

$$\mathcal{F}[\mathbf{S}_1,\mathbf{S}_2] = \mathcal{F}_p[\mathbf{S}_1,\mathbf{S}_2]\mathcal{F}_e[\mathbf{S}_1,\mathbf{S}_2]. \quad (97)$$

The functional associated with the electrons is approximated by the exponential of the effective action of Eq. (62) and the one associated with the phonons is given by

$$\mathcal{F}_p[\mathbf{S},\mathbf{D}] = \exp\Bigg\{-\frac{i}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \times \sum_{\mathbf{k}} \left(\sqrt{2}S_\alpha(t_1,-\mathbf{k}) \;\; \frac{1}{\sqrt{2}}D_\alpha(t_1,-\mathbf{k})\right) \times \sigma_1 \tilde{G}_{\alpha\gamma}(t_1-t_2,\mathbf{k})\sigma_1 \begin{pmatrix} \sqrt{2}S_\gamma(t_2,\mathbf{k}) \\ \frac{1}{\sqrt{2}}D_\gamma(t_2,\mathbf{k}) \end{pmatrix}\Bigg\}, \quad (98)$$

where

$$\frac{i}{2}\tilde{G}_{\alpha\gamma}(t,\mathbf{k}) = \sum_\mu H^*_{\alpha\mu}(\mathbf{k})e^{-i\omega_\mu(\mathbf{k})t} \times \begin{pmatrix} 1+2n(\omega_\mu(\mathbf{k})) & \theta(t) \\ -\theta(-t) & 0 \end{pmatrix} H_{\gamma\mu}(\mathbf{k}) = \begin{pmatrix} (\Lambda_\beta)_{\alpha\gamma}(t,\mathbf{k}) & (\Lambda)_{\alpha\gamma}(t,\mathbf{k})\theta(t) \\ -(\Lambda)_{\alpha\gamma}(t,\mathbf{k})\theta(-t) & 0 \end{pmatrix}. \quad (99)$$

Regarding the introduction of a random field into the equations of motion, now its correlations are given by the sum of two terms. The one arising from the electrons was already calculated in Sec. III. The contribution from the phonons is given by

$$\langle \xi_\alpha(t,\mathbf{x})\xi_\beta(0)\rangle_{\text{phonons}} = \sum_{\mathbf{k},\mu} e^{i[\mathbf{k}\cdot\mathbf{x}-\omega_\mu(\mathbf{k})t]} \times H^*_{\alpha\mu}(\mathbf{k})H_{\beta\mu}(\mathbf{k})\coth\left(\frac{\beta\omega_\mu(\mathbf{k})}{2}\right). \quad (100)$$

Assuming that $H_{\alpha\mu}(\mathbf{k}) = H_\alpha(\omega_\mu(\mathbf{k}))$, this last expression can be written as

$$\langle \xi_\alpha(t,\mathbf{x})\xi_\beta(0)\rangle_{\text{phonons}} = \int_0^\infty d\omega \rho(\omega,\mathbf{x})e^{-i\omega_\mu(\mathbf{k})t} \times H^*_\alpha(\omega_\mu(\mathbf{k}))H_\beta(\omega_\mu(\mathbf{k}))\coth(\frac{\beta\omega}{2}), \quad (101)$$

in which

$$\rho(\omega,\mathbf{x}) = \sum_{\mathbf{k},\mu}\delta(\omega-\omega_\mu(\mathbf{k}))e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (102)$$

If we make the following additional assumption that

$$\rho(\omega,\mathbf{x}) = \rho(\omega)\delta^3(\mathbf{x}), \quad (103)$$

then the discussion of the simplified model of Sec. II applies to this part of the correlation function and we have for the full correlation function

$$\text{Re}\{\langle \xi_\alpha(t,\mathbf{x})\xi_\beta(0)\rangle\} = \delta_{\alpha\beta}(\alpha_G\varphi(t)\delta^3(\mathbf{x}) + \frac{\lambda^2}{2}\text{Re}\{[\mathcal{F}^{-1}g](t,\mathbf{x})\}), \quad (104)$$

in which we have applied the considerations we made in Sec. II regarding the coupling constants. The discussion of the validity of the fluctuation-dissipation theorem is the same as in the end of Secs. II and III. It is clearly satisfied by both contributions.

V. CONCLUSIONS

Even though the model considered is very complex, we were able to obtain very interesting results. In the limit of high temperatures, the magnetic moments associated to the $3d$ electrons feel a random field which has two contributions: one from the interaction with the phonons which, with some additional assumptions regarding coupling constants, can be reduced to the one predicted by Brown;[11] and the other one which comes from the interaction with the conduction electrons. In addition to an external magnetic field, the magnetic moments also feel an effective magnetic field associated with their interaction with electrons. This effective magnetic field, given by Eq. (B13), is related to the magnetic field felt by the electrons, $\mathbf{h}(\mathbf{k})$, and explicitly manifests the Fermi-Dirac statistics of these particles because it is weighted by the function $(\mathcal{S}g)(t,\mathbf{k}) = (1/2)[\sum_{\mathbf{k}'}(1 - f(\epsilon(\mathbf{k}')))f(\epsilon(\mathbf{k}' - \mathbf{k}))e^{-i[\epsilon(\mathbf{k}')-\epsilon(\mathbf{k}'-\mathbf{k})]t} - (t \to -t,\mathbf{k} \to -\mathbf{k})]$, where f is the Fermi-Dirac distribution. In the limit in which the interaction with conduction electrons is small compared with the interaction with phonons, $\max_\mu |\gamma_\mu|/\lambda \gg 1$, one can obtain the Landau-Lifshitz-Bloch equation from the interaction with phonons using the results of Rebei *et al.*[9] and Garanin,[14] which is a remarkable fact.

The limit of high temperatures should be further investigated in the case of the contribution given by the conduction electrons since it is not clear from the expression of the random field correlation function that it will have Brown's form, i.e., white noise, under some assumption of the conduction electrons' energy spectrum and density of states in such a limit. If this is the case, then the Gilbert damping constant is given by the sum of two terms, one from the phonons and the other from the electrons. The one given by the conduction electrons appears to be independent of the electron density of states and is equal to $\lambda^2/16$.

The case of finite temperatures is described by two generalized functions which are essentially the Fourier transforms of some given functions characterizing the type of interaction, the statistics, and the density of states of the bath degrees of freedom.

The part of the correlation function of the random field which comes from the interaction with the phonons yields an intimate relation between friction (the associated Gilbert constant) and the random field fluctuations; see Eq. (32). The validity of the Markovian approximation is measured in terms of the thermal correlation time τ_{th}. For time scales shorter than this, the Markovian approximation for the stochastic field fails. In the case of the contribution for the random field correlation function given by the electrons, we see that Eq. (84) manifests the existence of a fluctuation-dissipation theorem. This is consistent with the expansion done being simply a linear-response theory. Thus the theory considered here satisfies a general fluctuation-dissipation theorem which relates the random field fluctuations to the friction constants which measure the effect of the interaction of the spin with electrons and phonons.

The electron contribution to the noise correlation functions exhibits the properties of the Fermi level of the electrons. These properties are better seen at low temperatures in which Friedel oscillations appear naturally (see Fig. 2). By measuring noise correlation functions one is indeed probing the Fermi surface of the ferromagnet in study. We have also shown that at any temperature the correlation decays as $\sim\sqrt{\mu T}$, meaning that one can also probe the effective mass, μ, for the conduction electrons by measuring this correlation function at different temperatures. This shows that, by studying the noisy dynamics of impurities in a ferromagnet (represented by a spin-j field), one can extract physical properties of the ferromagnetic material.

Our model is more general than the so-called three-temperature model considered and validated in experiments of Beaurepaire *et al.*[17] This is because we do not assume that the spins, electrons, and phonons are thermalized. Instead, we consider that at an initial time t_0 they are thermalized and the density matrix decouples at that time, but the time evolution couples the various systems and, thus, considering individual temperatures at each instant has no precise meaning within this model.

We believe that the proposed model can lead to a correct qualitative description of the ultrafast dynamics of magnetic moments in the case of laser-induced excitations at time scales shorter than the picosecond.

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APPENDIX A: EFFECTIVE ACTION FOR THE SPIN-PHONON THEORY

Following Rebei and Parker,[9] we consider now the reduced density matrix of the system in the model of Sec. II,

$$\hat{\rho}_S(t) = \text{Tr}_R\{\hat{\mathcal{U}}(t;t_0)\rho(t_0)\hat{\mathcal{U}}^\dagger(t;t_0)\}, \tag{A1}$$

where $\hat{\mathcal{U}}(t;t_0) = T_t \exp(-i\int_{t_0}^t \hat{\mathcal{H}}(s)ds)$ and T_t is the time ordering operator.

We assume that at $t = t_0$ the spin and bath are decoupled, and the bath is in an equilibrium state with temperature T. Then we have

$$\begin{aligned} \hat{\rho}(t_0) &= \hat{\rho}_S(t_0) \otimes \hat{\rho}_R(t_0), \\ \hat{\rho}_R(t_0) &= Z_R^{-1}\exp(-\beta\hat{\mathcal{H}}_R), \end{aligned} \tag{A2}$$

where $Z_R = \text{Tr}\{\exp(-\beta\hat{\mathcal{H}}_R)\}$ is the partition function for the bath, and we denoted $\beta = 1/k_B T$.

For the reduced density matrix we choose a path-integral representation using the basis of coherent states in the Hilbert space of the physical system. For the phonons we use the holomorphic representation in which the coherent states are defined by

$$||\alpha\rangle = \prod_{\mathbf{k}} \exp(\hat{a}_\mu^\dagger(\mathbf{k})\alpha_\mu(\mathbf{k}))|0\rangle, \tag{A3}$$

which satisfies the following property:

$$\hat{a}_\mu(\mathbf{k})||\alpha\rangle = \alpha_\mu(\mathbf{k})||\alpha\rangle. \quad \text{(A4)}$$

These states provide a decomposition of the identity (of the reservoir)

$$\hat{\mathbb{1}}_R = \int d\mu(\alpha)||\alpha\rangle\langle\alpha||,$$
$$d\mu(\alpha) = \prod_{\mathbf{k},\mu} \frac{d^2\alpha_\mu(\mathbf{k})}{\pi} e^{-\alpha^*_\mu(\mathbf{k})\alpha_\mu(\mathbf{k})}. \quad \text{(A5)}$$

For the spin j we use the spin coherent states,[30–32]

$$|\mathbf{S}\rangle = (1+|\zeta(\mathbf{S})|^2)^{-j}\exp(\zeta(\mathbf{S})\hat{S}_-)|jj\rangle, \quad \text{(A6)}$$

with

$$\zeta(\mathbf{S}) = e^{i\varphi}\tan\frac{\theta}{2}, \quad \text{(A7)}$$

$$\mathbf{S} = (S_\alpha) = j\,(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^T, \quad \text{(A8)}$$

where $\hat{S}_\pm = \hat{S}_1 \pm i\hat{S}_2$ and $|jj\rangle$ is the highest weight vector of the representation of the SU(2) group labeled by j. These states have the property

$$\langle\mathbf{S}|\hat{S}_\alpha|\mathbf{S}\rangle = S_\alpha \quad \text{(A9)}$$

and provide a decomposition of the identity operator in the form

$$\hat{\mathbb{1}}_S = \int d\mu(\mathbf{S})\,|\mathbf{S}\rangle\,\langle\mathbf{S}|\,,$$
$$d\mu(\mathbf{S}) = \frac{2j+1}{4\pi}\delta(\mathbf{S}^2 - j^2)d^3S. \quad \text{(A10)}$$

The properties (A4) and (A9) allow us to establish a correspondence between classical and quantum quantities. We consider path-integral representations which make use of coherent-state matrix elements which can be computed trivially, using Eq. (A4) and its Hermitian conjugate, and Eq. (A9), for linear terms in the spin operators. A general prescription to obtain (spin) coherent-state matrix elements of operators is given in Ref. 30, where a holomorphic definition of the (spin) coherent states is used.

Matrix elements of the reduced density matrix $\hat{\rho}(t)$ with the spin coherent states are

$$\rho_S(\mathbf{S}_f,\mathbf{S}_i,t) = \langle\mathbf{S}_f|\hat{\rho}_S(t)|\mathbf{S}_i\rangle$$
$$= \langle\mathbf{S}_f|\mathrm{Tr}_R\{\hat{\mathcal{U}}(t;t_0)\rho(t_0)\hat{\mathcal{U}}^\dagger(t;t_0)\}|\mathbf{S}_i\rangle. \quad \text{(A11)}$$

They are related to the density matrix at time $t = 0$ through the propagator $J(\mathbf{S}_f,\mathbf{S}_i,t;\mathbf{S}_2,\mathbf{S}_1,t_0)$, given by

$$\rho_S(\mathbf{S}_f,\mathbf{S}_i,t) = \int d\mu(\mathbf{S}_1)d\mu(\mathbf{S}_2)J(\mathbf{S}_f,\mathbf{S}_i,t;\mathbf{S}_2,\mathbf{S}_1,t_0)$$
$$\times\,\rho_S(\mathbf{S}_1,\mathbf{S}_2,t_0). \quad \text{(A12)}$$

The propagator has a path-integral representation of the form

$$J(\mathbf{S}_f,\mathbf{S}_i,t;\mathbf{S}_2,\mathbf{S}_1,t_0)$$
$$= \int_{\mathbf{S}_1}^{\mathbf{S}_f}\mathcal{D}\mu(\mathbf{S}'_1)\int_{\mathbf{S}_2}^{\mathbf{S}_i}\mathcal{D}\mu(\mathbf{S}'_2)\exp(iI[\mathbf{S}'_1,\mathbf{S}'_2])\mathcal{F}[\mathbf{S}'_1,\mathbf{S}'_2], \quad \text{(A13)}$$

where

$$I[\mathbf{S}_1,\mathbf{S}_2] := S_{\mathrm{WZ}}[\mathbf{S}_1] - S_{\mathrm{WZ}}[\mathbf{S}_2]$$
$$+ S_S[\mathbf{S}_1] - S_S[\mathbf{S}_2], \quad \text{(A14)}$$

in which S_{WZ} is the Wess-Zumino action,

$$S_{\mathrm{WZ}}[\mathbf{S}] = j\int_{t_0}^{t}dt\int_0^1 du\;\mathbf{n}\cdot(\partial_u\mathbf{n}\times\partial_t\mathbf{n}), \quad \text{(A15)}$$

where

$$\mathbf{n}(u,t) = (\sin(u\theta)\cos\varphi,\ \sin(u\theta)\sin\varphi,\ \cos(u\theta))^T \quad \text{(A16)}$$

is a map which continuously deforms the constant curve given by $\mathbf{n}_0(t) = \mathbf{n}_0 = (0,0,1)^T$ (the north pole) into the curve $\mathbf{n}(t) = \mathbf{S}(t)/j$ through a great circle, a geodesic, in S^2. We see that the Wess-Zumino action, given by Eq. (A15), gives the area enclosed by the path traced by the spin vector and the two great circles from the north pole of the sphere to the endpoints of that path. We also have

$$S_S[\mathbf{S}] = -\int_{t_0}^{t}dt\,\langle\mathbf{S}(t)|\hat{\mathcal{H}}_S(t)|\mathbf{S}(t)\rangle = \int_{t_0}^{t}dt\;\mathbf{S}(t)\cdot\mathbf{H}, \quad \text{(A17)}$$

and $\mathcal{F}[\mathbf{S}_1,\mathbf{S}_2]$ is the Feynman-Vernon influence functional which can be written as

$$\mathcal{F}[\mathbf{S}_1,\mathbf{S}_2] = \int d\mu(\alpha_0)d\mu(\alpha_1)d\mu(\alpha_2)\;k(\alpha_0^\dagger,t;\alpha_1,t_0|\mathbf{S}_1)$$
$$\times\,Z_R^{-1}k(\alpha_1^\dagger,-i\beta;\alpha_2,0|0)\times k^*(\alpha_0^\dagger,t;\alpha_2,t_0|\mathbf{S}_2), \quad \text{(A18)}$$

where we defined the kernel

$$k(\alpha_f^\dagger,t_f;\alpha_i,t_i|\mathbf{S}) = \int\mathcal{D}^2\alpha\,\exp\left[\frac{1}{2}(\alpha_f^\dagger\alpha(t_f)+\alpha(t_i)^\dagger\alpha_i)\right]$$
$$\times\,\exp\left[i\int_{t_i}^{t_f}dt\,L\right], \quad \text{(A19)}$$

in which

$$iL = \frac{1}{2}\left(\frac{d\alpha^\dagger}{dt}\alpha - \alpha^\dagger\frac{d\alpha}{dt}\right) - i[\mathcal{H}_R(\alpha^\dagger(t),\alpha(t))$$
$$+\,\mathcal{H}_i(\alpha^\dagger(t),\alpha(t),\mathbf{S}(t))] \quad \text{(A20)}$$

and

$$\mathcal{H}_R = \frac{\langle\alpha(t)||\hat{\mathcal{H}}_R(t)||\alpha(t)\rangle}{\langle\alpha(t)||\hat{\mathbb{1}}_R||\alpha(t)\rangle}, \quad \text{(A21)}$$

$$\mathcal{H}_i = \frac{\langle\mathbf{S}(t)|\langle\alpha(t)||\hat{\mathcal{H}}_i(t)||\alpha(t)\rangle|\mathbf{S}(t)\rangle}{\langle\alpha(t)||\hat{\mathbb{1}}_R||\alpha(t)\rangle}. \quad \text{(A22)}$$

Note that in the formulas presented above we adopted vector notations $\alpha = (\alpha_\mu(\mathbf{k}))$ and $\alpha^\dagger = (\alpha^*_\mu(\mathbf{k}))$.

The Feynman-Vernon functional can be computed exactly because the integrals are Gaussian. After some algebra we obtain

$$\mathcal{F}[\mathbf{S}_1,\mathbf{S}_2] = \exp\left\{-i\left[\int_{t_0}^{t}\int_{t_0}^{t}dt_1dt_2(S_{1,\alpha}(t_1)\;S_{2,\alpha}(t_1))\right.\right.$$
$$\left.\left.\times\,\sigma_3\hat{G}_{\alpha\beta}(t_1-t_2)\sigma_3\begin{pmatrix}S_{1,\beta}(t_2)\\ S_{2,\beta}(t_2)\end{pmatrix}\right]\right\}, \quad \text{(A23)}$$

where

$$i\hat{G}_{\alpha\beta}(t) = \sum_{\mathbf{k},\mu} H^*_{\alpha\mu}(\mathbf{k})e^{-i\omega_\mu(\mathbf{k})t}\begin{pmatrix} (n(\omega_\mu(\mathbf{k}))+1)\theta(t)+n(\omega_\mu(\mathbf{k}))\theta(-t) & n(\omega_\mu(\mathbf{k})) \\ n(\omega_\mu(\mathbf{k}))+1 & (n(\omega_\mu(\mathbf{k}))+1)\theta(-t)+n(\omega_\mu(\mathbf{k}))\theta(t)\end{pmatrix} H_{\beta\mu}(\mathbf{k}), \tag{A24}$$

in which $n(\omega) = (e^{\beta\omega}-1)^{-1}$ is the Bose-Einstein distribution function.

We can introduce the fields associated with the Keldysh representation

$$\begin{pmatrix} S_\alpha(t) \\ D_\alpha(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(S_{\alpha,+}(t)+S_{\alpha,-}(t)) \\ S_{\alpha,+}(t)-S_{\alpha,-}(t) \end{pmatrix}, \tag{A25}$$

where the field $\mathbf{S}(t)$ is associated with the classical spin and $\mathbf{D}(t)$ with quantum and thermal fluctuations. The Feynman-Vernon functional in terms of these fields is

$$\mathcal{F}[\mathbf{S},\mathbf{D}] = \exp\left\{-i\int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \left(\sqrt{2}S_\alpha(t_1)\ \frac{1}{\sqrt{2}}D_\alpha(t_1)\right) \times \sigma_1 \tilde{G}^{\alpha\beta}(t_1-t_2)\sigma_1 \begin{pmatrix} \sqrt{2}S_\beta(t_2) \\ \frac{1}{\sqrt{2}}D_\beta(t_2)\end{pmatrix}\right\}, \tag{A26}$$

where

$$i\tilde{G}_{\alpha\beta}(t) = \sum_{\mathbf{k},\mu} H^*_{\alpha\mu}(\mathbf{k})e^{-i\omega_\mu(\mathbf{k})t} \times \begin{pmatrix} 1+2n(\omega_\mu(\mathbf{k})) & \theta(t) \\ -\theta(-t) & 0 \end{pmatrix} H_{\beta\mu}(\mathbf{k}). \tag{A27}$$

In the stationary phase approximation we take the variation of the action phase in the path integral to be zero. The corresponding equations of motion are

$$\dot{\mathbf{D}}(t) = \mathbf{S}(t)\times\frac{\delta S_{\text{eff}}}{\delta\mathbf{S}(t)} + \mathbf{D}(t)\times\frac{\delta S_{\text{eff}}}{\delta\mathbf{D}(t)}, \tag{A28}$$

$$\dot{\mathbf{S}}(t) = \mathbf{S}(t)\times\frac{\delta S_{\text{eff}}}{\delta\mathbf{D}(t)} + \frac{1}{4}\mathbf{D}(t)\times\frac{\delta S_{\text{eff}}}{\delta\mathbf{S}(t)}, \tag{A29}$$

where

$$S_{\text{eff}}[\mathbf{S},\mathbf{D}] = \int_{t_0}^t dt\ \mathbf{D}(t)\cdot\mathbf{H} - i\log\mathcal{F}[\mathbf{S},\mathbf{D}]. \tag{A30}$$

APPENDIX B: EQUATIONS OF MOTION

The equations of motion in momentum space of the full theory resulting from taking the variation of the effective action (in which we consider the truncated expansion of Sec. III) appearing in the path-integral representation of the spin density matrix are

$$\dot{D}(s,\mathbf{k}) = \sum_{\mathbf{p}}[\mathbf{S}(s,\mathbf{k}-\mathbf{p})\times\mathbf{W}(s,\mathbf{p}) + \mathbf{D}(s,\mathbf{k}-\mathbf{p}) \times (\mathbf{T}(s,\mathbf{p})+\mathbf{H}_{\text{eff}}(s,\mathbf{p}))], \tag{B1}$$

$$\dot{S}(s,\mathbf{k}) = \sum_{\mathbf{p}}\Big[\mathbf{S}(s,\mathbf{k}-\mathbf{p})\times(J\mathbf{k}^2\mathbf{S}(s,\mathbf{p})+\mathbf{T}(s,\mathbf{p})+\mathbf{H}_{\text{eff}}(s,\mathbf{p})) + \frac{1}{4}\mathbf{D}(s,\mathbf{k}-\mathbf{p})\times(J\mathbf{k}^2\mathbf{D}(s,\mathbf{p})+\mathbf{W}(s,\mathbf{p}))\Big], \tag{B2}$$

in which

$$\mathbf{T}(s,-\mathbf{k}) = \mathbf{T}^{(S)}(s,-\mathbf{k}) + \mathbf{T}^{(D)}(s,-\mathbf{k}), \tag{B3}$$

$$\mathbf{T}^{(S)}(s,-\mathbf{k}) = \mathbf{T}^{(S)}_{\text{ph}}(s,-\mathbf{k}) + \mathbf{T}^{(S)}_{\text{sd}}(s,-\mathbf{k}), \tag{B4}$$

$$\mathbf{T}^{(D)}(s,-\mathbf{k}) = \mathbf{T}^{(D)}_{\text{ph}}(s,-\mathbf{k}) + \mathbf{T}^{(D)}_{\text{sd}}(s,-\mathbf{k}), \tag{B5}$$

$$\mathbf{T}^{(S)}_{\text{ph}}(s,-\mathbf{k}) = i\int_{t_0}^t ds'[\Lambda(s-s',-\mathbf{k})-\Lambda(s'-s,\mathbf{k})] \times\theta(s-s')\mathbf{S}(s',-\mathbf{k}), \tag{B6}$$

$$\mathbf{T}^{(S)}_{\text{sd}}(s,-\mathbf{k}) = i\lambda^2\int_{t_0}^t ds'(\mathcal{S}g)(s-s',-\mathbf{k}) \times\theta(s-s')\mathbf{S}(s',-\mathbf{k}), \tag{B7}$$

$$\mathbf{T}^{(D)}_{\text{ph}}(s,-\mathbf{k}) = i\int_{t_0}^t ds'[\Lambda_\beta(s-s',-\mathbf{k})+\Lambda_\beta(s'-s,\mathbf{k})] \times\mathbf{D}(s',-\mathbf{k}), \tag{B8}$$

$$\mathbf{T}^{(D)}_{\text{sd}}(s,-\mathbf{k}) = i\frac{\lambda^2}{2}\int_{t_0}^t ds'(\mathcal{P}g)(s-s',\mathbf{k})\mathbf{D}(s',-\mathbf{k}), \tag{B9}$$

$$\mathbf{W}(s,-\mathbf{k}) = \mathbf{W}_{\text{ph}}(s,-\mathbf{k}) + \mathbf{W}_{\text{sd}}(s,-\mathbf{k}), \tag{B10}$$

$$\mathbf{W}_{\text{ph}}(s,-\mathbf{k}) = -i\int_{t_0}^t ds'[\Lambda(s-s',-\mathbf{k})-\Lambda(s'-s,\mathbf{k})] \times\theta(s'-s)\mathbf{D}(s',-\mathbf{k}), \tag{B11}$$

$$\mathbf{W}_{\text{sd}}(s,-\mathbf{k}) = -i\lambda^2\int_{t_0}^t ds'(\mathcal{S}g)(s-s',-\mathbf{k})\theta(s'-s) \times\mathbf{D}(s',-\mathbf{k}), \tag{B12}$$

$$\mathbf{H}_{\text{eff}}(s,-\mathbf{k}) = \mathbf{H}(-\mathbf{k}) + i\lambda\int_{t_0}^t ds'(\mathcal{S}g)(s-s',\mathbf{k})\theta(s-s') \times\mathbf{h}(-\mathbf{k}'), \tag{B13}$$

where the indices "ph" and "sd" emphasize that these terms come from the interaction with phonons or with electrons, respectively. In the above expression the long-wavelength approximation has been taken for the Heisenberg exchange term in the action so that it becomes a diffusive term in the equation of motion in which J is some constant measuring the interaction strength.

APPENDIX C: COMPUTATION OF FREE THERMAL ELECTRON PROPAGATORS

Consider $t > 0$ and let us compute

$$\mathcal{G}^0(\mathbf{x},-\tau=-it)=\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}e^{i\epsilon(\mathbf{k})t}f(\epsilon(\mathbf{k})). \tag{C1}$$

This can be done by writing the sum on $\mathbf{k}$ as an integral ($\sum_{\mathbf{k}}\to\int d^3k/(2\pi)^3$),

$$\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}e^{i\epsilon(\mathbf{k})t}f(\epsilon(\mathbf{k})). \tag{C2}$$

Now we consider that the conduction electrons have a quadratic spectrum of the form $\epsilon(\mathbf{k})=\mathbf{k}^2/2\mu$, where μ is an effective mass. Using spherical coordinates, we see that the angular integrals are trivially computed and we end up with

$$-\frac{1}{(2\pi)^2r}\frac{d}{dr}\int_{-\infty}^{\infty}dk\frac{\exp[i(kr+\epsilon(k)t)]}{1+\exp[\beta\epsilon(k)]}. \tag{C3}$$

To compute

$$\int dk\frac{e^{i[kr+\epsilon(k)t]}}{1+e^{\beta\epsilon(k)}}, \tag{C4}$$

we complexify the integral with the contour in the upper half-plane being a semicircle, with radius $R\to\infty$, containing the poles. The poles are given by solving the algebraic equation

$$\beta\epsilon(k_n)=(2n+1)i\pi,\ n\in\mathbb{Z}. \tag{C5}$$

Near each pole,

$$\exp\left(\frac{\beta k^2}{2\mu}\right)\approx-\frac{\beta}{\mu}k_n(k-k_n)-1. \tag{C6}$$

The solution is

$$\int dk\frac{e^{i[kr+\epsilon(k)t]}}{1+e^{\beta\epsilon(k)}}=-\frac{2\pi i\mu}{\beta}\sum_{n=-\infty}^{\infty}{}'(k_n)^{-1}e^{i[k_nr+\epsilon(k_n)t]}, \tag{C7}$$

where in the summation the prime is there to indicate that we are only summing for values of k_n which have a positive imaginary part. Plugging this back into the expression for the Green's function yields

$$\mathcal{G}^0(\mathbf{x},-it)=-\frac{\mu}{2\pi r\beta}\sum_{n=-\infty}^{\infty}{}'e^{i[k_nr+\epsilon(k_n)t]}, \tag{C8}$$

where the sum is restricted to those k_n satisfying Eq. (C5), or

$$\frac{\beta k_n^2}{2\mu}=i(2n+1)\pi,\ n\in\mathbb{Z} \tag{C9}$$

having a positive imaginary part. Solving this equation gives

$$k_n=\pm(1+i)\sqrt{\frac{(2n+1)\mu\pi}{\beta}}. \tag{C10}$$

Suppose $2n+1>0$; then the solution with a positive imaginary part is given by

$$k_n=(1+i)\sqrt{\frac{(2n+1)\mu\pi}{\beta}}; \tag{C11}$$

now if $2n+1<0$, then

$$k_n=(-1+i)\sqrt{\frac{|2n+1|\mu\pi}{\beta}}. \tag{C12}$$

For fixed $|2n+1|$ we have the two contributions

$$e^{\left[i(1+i)\sqrt{\frac{|2n+1|\mu\pi}{\beta}}r+i\times i|\epsilon(k_n)|t\right]}+e^{\left[i(-1+i)\sqrt{\frac{|2n+1|\mu\pi}{\beta}}r-i\times i|\epsilon(k_n)|t\right]}, \tag{C13}$$

which gives, after simple algebra,

$$2e^{-\sqrt{\frac{|2n+1|\mu\pi}{\beta}}r}\cos\left[\sqrt{\frac{|2n+1|\mu\pi}{\beta}}r+i|2n+1|\pi t\right]. \tag{C14}$$

So the sum becomes

$$\mathcal{G}^0(\mathbf{x},-it)=-\frac{\mu}{\pi r\beta}\sum_{n=0}^{\infty}e^{-\sqrt{\frac{(2n+1)\mu\pi}{\beta}}r}\times\cos\left[\sqrt{\frac{(2n+1)\mu\pi}{\beta}}r+i(2n+1)\pi t\right]. \tag{C15}$$

The other inverse Fourier transform gives

$$\begin{aligned}-\mathcal{G}^0(|-\mathbf{x}|=|\mathbf{x}|=r,it)&=\left(\frac{\mu i}{2\pi t}\right)^{3/2}\exp\left(\frac{ir^2\mu}{2t}\right)\\&\quad+\frac{\mu}{\pi r\beta}\sum_{n=0}^{\infty}e^{-\sqrt{\frac{(2n+1)\mu\pi}{\beta}}r}\\&\quad\times\cos\left[\sqrt{\frac{(2n+1)\mu\pi}{\beta}}r-i(2n+1)\pi t\right],\end{aligned} \tag{C16}$$

where the first term comes from the (inverse) Fourier transform of the exponential of the conduction-band energy, which is a Gaussian integral.

*bruno.mera@ist.utl.pt

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