Chiral classical states in a rhombus and a rhombi chain of Josephson junctions with two-band superconducting elements

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We present a study of Josephson junctions arrays with two-band superconducting elements in the high-capacitance limit. We consider two particular geometries for these arrays: a single rhombus and a rhombi chain with two-band superconducting elements at the spinal positions. We show that the rhombus shaped JJ circuit and the rhombi chain can be mapped onto a triangular JJ circuit and a JJ two-leg ladder, respectively, with zero effective magnetic flux, but with Josephson couplings that are magnetic flux dependent. If the two-band superconductors are in a sign-reversed pairing state, one observes transitions to or from chiral phase configurations in the mapped superconducting arrays when magnetic flux or temperature are varied. The phase diagram for these chiral configurations is discussed. When half-flux quantum threads each rhombus plaquette, new phase configurations of the rhombi chain appear that are characterized by the doubling of the periodicity of the energy density along the chain, with every other two-band superconductor locked in a sign-reversed state. In the case of identical Josephson couplings, the energy of these phase configurations becomes independent of the inner flux in the rhombi chain and the supercurrent along the rhombi chain is zero.

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I. INTRODUCTION

Frustrated Josephson junction (JJ) arrays have been extensively studied in the last decades [1–8]. These studies have been to a large extent motivated by the close analogy with frustrated classical spin systems [9–11] and more recently by the possible relevance of these systems to quantum computation [5,6,12]. This analogy requires the conditions of low temperature and symmetric JJs, both in experimental and theoretical studies. Such conditions imply that the superconducting phases are the only variables required to describe the superconducting system. If the Josephson current is studied as a function of temperature, the variation of the magnitudes of the superconducting order parameters has to be taken into account, particularly when asymmetric junctions are present or if, as in the case of the present paper, two-band superconducting elements are introduced in the JJ array.

In the case of a JJ array with two-band superconducting elements, besides the usual Josephson tunnelings, one must consider the interband tunnelings [13,14], which effectively modify the geometry of the JJ array. In fact, a multiband superconductor can be regarded as a simple realization of a short JJ array. The interband pairings are equivalent to Josephson tunnelings and, for example, a three-band superconductor is analogous to a triangular circuit of asymmetric JJs [15–18]. Frustration in this case occurs if one or several interband interactions are repulsive since a repulsive interband interaction in a multiband superconductor plays a similar role to that of a π junction in a Josephson junction array [19]. Another superconducting system equivalent to a triangular JJ array is a single JJ between a two-band superconductor (TBS) and a single-band s-wave superconductor. In this system, chiral states have also been predicted in the absence of magnetic flux if the TBS is in a sign-reversed state [13,16]. Such a sign-reversed two-band scenario, the so-called $s_\pm$ state, has been recently proposed in the context of the iron-based superconductors [20].

The JJ rhombi chain of one-band superconductors (OBS) is one of the most simple geometries where frustration effects have been studied theoretically and experimentally [4,5,7,8]. It has been shown that in this system, the frustration induced by a magnetic flux of half-flux quantum per plaquette leads to the halving of the period of the supercurrent sawtooth profile obtained with variation of the magnetic flux in the chain. In this manuscript, we present a study of a rhombus and a rhombi chain with two-band (one-band) superconducting elements at the spinal (edge) positions in the high-capacitance limit (see Figs. 1 and 2). We assume a closed ring geometry for the rhombi chain [7,21] and magnetic flux threads the ring as well as each of the rhombi plaquettes (see Fig. 1).

One of the main results of this paper is the mapping of the rhombus and the rhombi chain onto a triangular JJ circuit and a JJ two-leg ladder, respectively, with zero effective magnetic flux in each plaquette, but with Josephson couplings that are magnetic flux dependent. If the TBSs are in a sign-reversed pairing state, transitions are observed between chiral and nonchiral states in the effective JJ arrays induced by magnetic field. We show that the existence of a chiral state in the effective JJ arrays depends on the relative signs of the interband couplings and Josephson tunneling constants, on their absolute values and on the value of magnetic flux, and the respective phase diagram is constructed.

A second important result is that, as the magnetic flux approaches half-flux quantum per rhombus plaquette, one
observes a halving of the period in the energy-phase plot as in the case of one-band rhombi chain [5], but with additional structure due to new phase configurations, which are characterized by the doubling of the periodicity of the energy density along the rhombi chain, with every other TBS locked in a sign-reversed state. If the system is completely symmetric in what concerns the Josephson couplings, the energy of these phase configurations becomes independent of the inner flux in the rhombi chain, and consequently, plateaus are observed in the energy-phase plots, which become wider as the Josephson tunneling constants are decreased. The respective supercurrent along the chain is therefore blocked and the usual sawtoothlike tunneling constants are decreased. The respective supercurrent along the chain is therefore blocked and the usual sawtoothlike tunneling constants are decreased. The respective supercurrent along the chain is therefore blocked and the usual sawtoothlike tunneling constants are decreased. The respective supercurrent along the chain is therefore blocked and the usual sawtoothlike tunneling constants are decreased.

The Hamiltonian of a linear JJ chain in the high-capacitance limit is given by (see Appendix A)

$$H = -\sum_i J \cos(\phi_i - \phi_{i+1} - A_{i,i+1}).$$  \hspace{1cm} (1)

If open boundary conditions are assumed (for instance, removing one JJ from a ring of JJs), the energy of each JJ can be minimized independently, that is, the argument of all cosines can be simultaneously zero, $\phi_i - \phi_{i+1} - A_{i,i+1} = 0$, \forall i, and the minimum energy is $-N_J f$, where $N_J$ is the number of JJs. If periodic boundary conditions are assumed (a ring of JJs), one could still try a similar approach, that is, one could impose that the argument of all cosines are zero except for the JJ, which was added to the previous JJ chain with open boundary, $\phi_I - A_{i,i+1} = 0$ for $i = 1, \ldots, N - 1$. This implies that $\phi_N - \phi_1 - A_{N-1} = -2\pi f$, where $\sum_{i=1}^N A_{i,i+1} = 2\pi f = 2\pi J f_0$, and therefore the respective Josephson energy becomes

$$E_J = -(N - 1)J - J \cos(\phi_I),$$  \hspace{1cm} (2)

where the magnetic flux $\Phi_i$ (and all fluxes in the remaining part of this paper) is written in units of $\Phi_0 / 2\pi$, that is, $\Phi_i = 2\pi$ corresponds to the magnetic flux quantum. However, this is not the energy absolute minimum of the JJ ring. This can be shown by mapping the JJ ring into a tight-binding Hamiltonian. The mapping is constructed using the fact that if one considers the two-site tight-binding Hamiltonian $H = -\epsilon_1 a_1^\dagger [1] |1\rangle - \epsilon_2 a_2^\dagger [2]$ and the state $|\psi\rangle = e^{i\Phi_1}|1\rangle + e^{i\Phi_2}|2\rangle$, then one has $\langle \psi | H | \psi \rangle = -2\cos(\phi_2 - \phi_1 + \pi_2)$. Thus the energy of the JJ ring is equivalent to the expectation value of the tight-binding Hamiltonian of a linear chain with periodic boundary conditions in the state $|\psi\rangle = \sum_{j=1}^N e^{i\phi_j}|j\rangle$. The eigenstates of the 1D tight-binding Hamiltonian are Bloch states $|k\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{ikj}|j\rangle$ where $k = n \cdot 2\pi / N$, with $n = 0, 1, \ldots, N - 1$ and we have defined the gauge such that $A_{i,i+1} = 2\pi f / N$, such that the tight-binding Hamiltonian is translationally invariant. The respective eigenvalues are $\epsilon_k = -2\cos(k - 2\pi f / N)$, independently of the gauge. The state $|\psi\rangle$ can be decomposed in the momentum basis $|k\rangle$ so that $|\psi\rangle = \sum_k a_k |k\rangle$ and $\langle \psi | H | \psi \rangle = \sum_k a_k^* \epsilon_k a_k$. If the case of the JJ ring, the correspondence $|\psi\rangle = \sqrt{N} |k\rangle$ is possible.
because $\sqrt{N}|k\rangle$ is a state of the form $\sum_{j=1}^{N} e^{i\phi_j}|j\rangle$. Therefore the minimum energy of the JJ chain is achieved selecting the $|\psi\rangle = \sqrt{N}|k_{\text{min}}\rangle$ state with the minimum energy which is $k_{\text{min}} = 0$ for $\Phi_i = 0$ and $k_{\text{min}} = -\text{int}(\Phi_i + \pi)/2\pi \cdot 2\pi/N$, for $\Phi_i \neq 0$. The respective energy is

$$E_J = -NJ \cos(k_{\text{min}} + \Phi_i/N),$$

which has a parabolic-like profile as a function of $\Phi_i$ with period $2\pi$. This energy is lower than the one in Eq. (2), but one should note that $N_f - 1$ junctions have a higher energy in this Bloch state, and it is the link that accumulates all the magnetic phase at the boundary [with energy $-J \cos(\Phi_i)$] that raises the total energy above the Bloch state energy. In the next section, we will show that in the case of the rhombi chain, the magnetic phase can be accumulated in a single rhombus with low energy cost, and this is why the halving of the periodicity of the supercurrent occurs.

B. Rhombi chain

The Hamiltonian of a rhombi chain of JJs of OBSs in the high-capacitance limit is given by

$$H = -J \sum \left[ J \cos(\phi^s_i - \phi^s_i - A^{\text{int}}_i) + J \cos(\phi^b_i - \phi^b_i - A^{\text{int}}_i) + J \cos(\phi^t_i - \phi^t_i - A^{\text{int}}_i) \right],$$

where $\phi^s_i$, $\phi^b_i$, $\phi^t_i$ are the phases at the spinal, top edge and bottom edge positions, respectively, and the index $i$ indicates the unit cell of the rhombi chain. The associated Peierls phases $A^{\text{int}}_i$ follow the same notation. The intraband energy contribution should be added to this Hamiltonian. This contribution determines the temperature dependence of the gap functions and consequently, of the Josephson couplings. This temperature can be introduced directly in the Josephson couplings of the previous Hamiltonian, using Eq. (B4). In this section, we assume a fixed temperature and ignore the constant intraband energy contribution. We also choose an uniform gauge such that $A^{\text{int}}_i = a$, $A^{\text{int}}_i = a$, $A^{\text{int}}_i = b$, and $A^{\text{int}}_i = b$, and therefore we have considered different phase shifts for the upper and lower parts of the rhombus, so that the normalized magnetic flux within the rhombus is $2a + 2b$. These phases are related with the phases acquired following the outer edge of the rhombi chain, $\Phi_o$, and following the inner edge, $\Phi_i$ (see Fig. 1 of Ref. [21]), so that

$$\Phi_o = 2aN_r,$$

$$\Phi_i = 2bN_r,$$

where $N_r$ is the number of rhombi. Note that the magnetic fluxes are written in units of $\Phi_0/(2\pi)$ so that they are numerically equal to the total Peierls phases in these closed paths. These global phases determine the average magnetic phase within the rhombi chain, $\Phi_1$, and the total phase within a rhombus, $\Phi_f$,

$$\Phi_1 = \frac{\Phi_o + \Phi_i}{2} = (a + b)N_r,$$

$$\Phi_f = \frac{\Phi_o - \Phi_i}{N_r} = 2a - 2b = 2\pi f.$$

In the case of the rhombi chain, the minimization can be performed in two steps: (i) minimization with respect to the edge phases for each rhombus with fixed spinal phases and (ii) minimization with respect to the spinal phases. Since the energy of one rhombus is

$$E_r = J \left[ \cos(\phi^s_1 - \phi^s_1 - a) + \cos(\phi^b_1 - \phi^b_1 - a) + \cos(\phi^t_1 - \phi^t_1 - a) + \cos(\phi^t_1 - \phi^t_1 - a) \right],$$

the minimization of $E_r$ with respect to the edge phases $\phi^s_1$ and $\phi^b_1$ (given the spinal phases $\phi^t_1$ and $\phi^t_1$) leads to

$$\tan(\phi^s_1 - \phi^s_1) = \tan(\phi^b_1 - \phi^b_1) = \tan\left(\frac{\Delta\phi}{2}\right),$$

where $\Delta\phi = \phi^b_1 - \phi^s_1$. Therefore the minimum energy of one rhombus is given by

$$\frac{E_r}{J} = -2 \left| \cos\left(a + \frac{\Delta\phi}{2}\right) - 2 \cos\left(b + \frac{\Delta\phi}{2}\right) \right|,$$

which can be rewritten as

$$E^\delta = -2J_\delta \cos\left(\frac{\Delta\Psi + \delta}{2}\right),$$

where $\Delta\Psi$ is the gauge invariant phase difference between the spinal nodes of the rhombus,

$$\Delta\Psi = \left(\frac{\Phi_1}{N_r} + \Delta\phi\right),$$

with renormalized Josephson couplings

$$J_\delta = 2J \left| \cos\left(\frac{\Phi_f}{4} - \frac{\delta}{2}\right) \right|$$

and where $\delta = 0$ or $\pi$, corresponding to solutions A and B in Fig. 3, respectively, whichever gives the minimum energy. These solutions reflect two possible chiral states in the rhombus for the given value of $\Delta\phi$. The previous energy can be interpreted as the one resulting from two linear JJs with the phase difference being equally distributed between the two junctions with an effective Josephson coupling $J$. For small $\Phi_1/2N_r$, $\Delta\phi$ and $\Phi_f$, one has $\delta = 0$ since this choice minimizes the phase in the cosine in Eq. (8) and maximizes

![FIG. 3. (Color online) Superconducting phases in a rhombus circuit of OBSs given a 2$\phi$ phase difference between spinal positions.](134513-3)
FIG. 4. (Color online) (a) Evolution of the normalized energy $\Delta E / \Delta E_{\text{max}} = (E - E_{\text{min}})/(E_{\text{max}} - E_{\text{min}})$ of the symmetric JJ rhombi array (with six rhombi) of OBSs as a function of magnetic flux $\Phi_f$, for several values of the frustrating magnetic field $\Phi_f$. (b) Example of supercurrent distribution in the rhombi chain for $\Phi_f = 0.97\pi$ and $\Phi_1 = \pi$. (c) Energies $E^i_1$ for $f = \Phi_f/(2\pi) = 0.482$ and $J = 1$ in a case of a chain with four rhombi. (d) Amplitude of the energy oscillations as $\Phi_f$ is varied ($E_{\text{max}} - E_{\text{min}}$ on the left axis), and minimum value of the energy ($E_{\text{min}}$ on the right axis) for the chain of six rhombi.

For $\Phi_f \approx \pi$, $J \approx 2J/\sqrt{2}$ for both possibilities, and the choice is dictated by the values of $\Phi_f / 2N_r$ and $\Delta \phi$.

One may now discuss the behavior of the rhombi chain as we did for the linear chain. Assuming $\Phi_1 = \pi$ and $\Phi_f \approx \pi$, the same comparison can be carried out between the Bloch state energy

$$E^i_1(\Phi_1) = -2N_r \tilde{J}_\delta \cos[k_{\min} + \Phi_1/(2N_r) - \delta/2],$$

(11)

with $\delta = 0$ or $\pi$, where $k_{\min} = 0$ for $\Phi_1 = 0$ and $k_{\min} = -\ln[(\Phi_1 + \pi)/2\pi]\pi/N_r$, for $\Phi_1 \neq 0$, and the energy of the configuration where all the magnetic phase is accumulated at the boundary rhombus

$$E^2_\delta(\Phi_1 = \pi) = -2(N_r - 1) \tilde{J}_\delta + 2 \tilde{J}_\delta \left| \cos\left(\frac{\pi + \delta}{2}\right) \right|.$$

(12)

Since $\tilde{J}_0 = \tilde{J}_\pi$ for $\Phi_f = \pi$, the energy minimum is obtained setting $\delta = \pi$ in the last term since that maximizes the absolute value of the cosine (in contrast to the linear chain result of the previous subsection) and therefore the energy becomes $-2N_r \tilde{J}_0$, which is also the energy of the rhombi chain when $\Phi_1 = 0$ (this reflects the halving of the periodicity of the energy as a function of $\Phi_1$ from $2\pi$ to $\pi$). This different choice of $\delta$ in one of the rhombi translates into a current flowing in the opposite direction to that of the other rhombi [see Fig. 4(b)]. Here, we have assumed that all the rhombi have $\delta = 0$ except one which has $\delta = \pi$, but one should also consider the configuration where all the rhombi have $\delta = \pi$ except one which has $\delta = 0$.

For general $\Phi_f$, the ground-state energy is the minimum between $E^i_1$ (with $\delta = 0$ or $\pi$) and the two energies associated with an inversion of the current in one rhombus,

$$E^2_\delta(\Phi_1) = -2[(N_r - 1) \tilde{J}_\delta + \tilde{J}_\delta \cos\left(\frac{\Delta \Phi}{2N_r}\right)],$$

(13)

where $\delta = 0$ and $\delta = \pi$ or $\delta = \pi$ and $\delta = 0$, and $\Delta \Phi$ is a periodic function of the flux $\Phi_f$: $\Delta \Phi = (\Phi_f (2\pi) - \pi$).

In Fig. 4(a), we show plots of the normalized energy $\Delta E / \Delta E_{\text{max}} = (E - E_{\text{min}})/(E_{\text{max}} - E_{\text{min}})$ of the symmetric JJ rhombi array (with six rhombi) of OBSs as a function of magnetic flux $\Phi_f$, for several values of the frustrating magnetic flux $\Phi_f$ close to half-flux quantum. The halving of the periodicity from $2\pi$ to $\pi$ is observed and as explained above, it is due to the relative lowering of the energies $E^\pi_1$ and $E^\pi_2$ in relation to $E^0_1$ and $E^0_2$. The relative positions of the energy curves are shown in Fig. 4(c) for $f = \Phi_f/(2\pi) = 0.482$ and $J = 1$ in the case of a chain with four rhombi. In Fig. 4(d), we show the amplitude of the energy oscillations as $\Phi_f$ is varied ($E_{\text{max}} - E_{\text{min}}$ on the left axis), and the minimum value of the energy ($E_{\text{min}}$ on the right axis) for the chain of six rhombi. The sharp reduction of the amplitude near $\Phi_f = \pi$ occurs because the energies $E^\pi_1$ and $E^\pi_2$ become lower than $E^0_1$ and $E^0_2$ in some
magnetic flux intervals [see Fig. 4(c)]. The interval where this sharp reduction occurs becomes narrower as the number of rhombi in the chain increases.

A curious feature of the rhombi chain is the dependence of normalized energy on the parity of the number of rhombi. For odd \(N_r\), the period of the energy as a function of \(\Phi_f\) is \(4\pi\) and not \(2\pi\). This reflects the fact that \(\Phi_f = 2\pi\) and \(N_r\) odd in our choice of gauge implies \(\Phi_1 = n\pi\) with \(n\) odd. This leads to a shift of \(\pi/2N_r\) (in the dependence of \(\Phi_1\)) of the curves of Fig. 4(a) when \(\Phi_f > \pi\).

The mapping into a linear chain can be generalized to more complex geometries, more precisely to 1D arrays of JJ clusters such that only one JJ connects a JJ cluster to the following one. One simple example is the linear chain plus a single rhombus, where similar behavior to that of Fig. 4 is observed, but modified by the fact that translation invariance is absent and that the previous mapping generates a linear chain with constant as well flux dependent Josephson couplings.

III. Rombus with two-band superconductors

The energy of an array of JJs with one-band and TBSs can also be described by the expression

\[
E = - \sum_{i,j} J_{ij} \cos(\phi_i - \phi_j - A_{ij}), \tag{14}
\]

where the index \(i\) labels the one-band superconducting elements as well as each of the bands of the TBSs (this implies that the interband coupling is interpreted as an additional Josephson coupling). The interband couplings and the Josephson constants are functions of the superconducting gaps and in order to find the minimum energy of the JJ array, one must minimize the sum of the previous energy and the superconducting condensation energy with respect to the absolute value of the superconducting gaps as well as with respect to the superconducting phases (see Appendix A1 for a more detailed discussion).

In the remaining of this paper, the magnitudes of the gap functions are determined taking into account the interband interactions but ignoring the Josephson tunneling between different superconductors (as usual for weak links). However, the superconducting phases are determined considering all the tunneling energy terms (interband tunneling and Josephson tunneling between different superconductors).

In this section, we study the possible superconducting states in the rhombus of Fig. 2, with two-band (one-band) superconducting elements at the spinal (edge) positions. The rhombus is assumed to be completely symmetric, so that the superconducting state of the rhombus is determined by the values of three couplings: the interband coupling \(J_{12}\) and the Josephson constants \(J_1\) and \(J_2\) associated with the tunnellings between the OBSs and each of the bands of the TBSs. We show below that this JJ circuit can be mapped into a triangular JJ circuit with effective Josephson couplings which again depend on the magnetic flux threading the rhombus.

In contrast to the case of a rhombus JJ circuit of OBSs, a chiral superconducting state can occur in the absence of magnetic field if the two-band superconductor is in a sign-reversed state or if one or more of the Josephson tunnellings constants is negative, that is, if a \(\pi\) junction is present. Note that this may be a partial \(\pi\) junction in the case of tunneling to TBSs, that is, the Josephson tunneling constant may only be negative for one of the bands.

The energy of a rhombus with spinal TBSs can be written as a sum of three terms (see Fig. 5),

\[
E_J = E_{J_1} + E_{J_2} + E_{12}, \tag{15}
\]

where \(E_{J_1}\) and \(E_{J_2}\) are the energies terms, which involve only the first or the second band of the TBSs, respectively, and which have exactly the same form of Eq. (5) (with \(b = -a\)),

\[
E_{J_1} = -J_1[\cos(\phi_1 - \phi_2 - a) + \cos(\phi_3 - \phi_4 - a) + \cos(\phi_5 - \phi_6 - a)] + \cos(\phi_3 - \phi_1 - a), \tag{16}
\]

\[
E_{J_2} = -J_2[\cos(\phi_4 - \phi_5 - a) + \cos(\phi_5 - \phi_6 - a) + \cos(\phi_6 - \phi_3 - a)] + \cos(\phi_3 - \phi_4 - a). \tag{17}
\]

The interband energy \(E_{12}\) is given by

\[
E_{12} = -J_{12}[\cos(\phi_1 - \phi_2) + \cos(\phi_5 - \phi_6)] \tag{18}
\]

and has no flux dependence. The flux threading the rhombus is given by \(\Phi_f = 2\pi f = 4a\).

If the interband coupling is zero and magnetic field is absent, all the Josephson energies can be simultaneously minimized and no frustration occurs. Two examples of nonchiral configurations are shown in Fig. 6. The introduction of the interband coupling may lead to frustration depending on the

FIG. 5. (Color online) The energy of a rhombus with spinal TBSs can be written as a sum of two terms, which involve only the first or the second band of the TBSs, and a third term with only the interband energy contribution.
The values of nodes in the presence of applied magnetic flux. The arrows exemplify the maximum energy state of a rhombus JJ circuit with TBSs at the spinal nodes, mapping the superconducting phases into XY spin angles. If the interband coupling is zero, all the Josephson energies can be simultaneously minimized and no frustration occurs. Here, we show the respective relative superconducting phase configurations, setting the superconducting phases into XY spin angles. 

Energies $E_{A,B}$ and $E_{C,D}$ are invariant under the transformations $(\phi \rightarrow -\phi, \alpha \rightarrow -\alpha)$ and $(\phi \rightarrow -\phi + \pi, \alpha \rightarrow -\alpha)$, respectively, which leads to additional configurations with the same energy as A and B or C and D. These complete the set of solutions found by numerical minimization of the rhombus energy. For $0 < \Phi_f < \pi$, the minimum energy is given by $E_{A,B}$. For $\pi < \Phi_f < 2\pi$, one finds $E_{C,D}$ as the minimum energy. So, the minimization of the energy of the rhombus has been reduced to finding the minimum energy of a triangular JJ circuit in the absence of flux with couplings $\tilde{J}_{13}$, $\tilde{J}_{23}$, and $\tilde{J}_{12}$ which depend on magnetic flux and with $\delta = 0$ when $0 < \Phi_f < \pi$ and $\delta = \pi$ when $\pi < \Phi_f < 2\pi$. This is one of the main results of this paper.

The previous expressions for the energy can be condensed in a single expression with an analogous form to Eq. (8),

$$E^{\delta} = -\tilde{J}_{13} \left| \cos \left( \frac{\Delta \Psi_i + \delta}{2} \right) \right| - \tilde{J}_{23} \left| \cos \left( \frac{\Delta \Psi_2 + \delta}{2} \right) \right| - \tilde{J}_{12} \cos(\alpha),$$

where $\Delta \Psi_i$ is the gauge invariant phase difference between the spinal nodes of the rhombus for band $i$ in the case of solutions A and C (note that $\Phi_1 = 0$, $\Delta \Psi_1 = \phi_5 - \phi_1 = 2\phi$ and $\Delta \Psi_2 = \phi_6 - \phi_2 = 2\phi - 2\alpha$. The couplings expressions can be equally rewritten as

$$\tilde{J}_{13} = 4J_{1i} \left| \cos \left( \frac{\Phi_f}{4} - \frac{\delta}{2} \right) \right|,$$

where $\delta$ is $0$ or $\pi$.

The triangular JJ circuit in the absence of magnetic flux has been studied in detail by Dias and Marques [15] in the context of frustrated three-band superconductivity [16–18]. In this circuit (recall that magnetic flux is absent), frustration arises if one or three of the Josephson couplings are negative ($\pi$ junctions or TBS with repulsive interband interaction), but the relative magnitudes of the couplings also affect the existence or absence of a chiral superconducting state. The minimization conditions of the energy of the triangular JJ circuit lead to nonfrustrated solutions $(\phi, \alpha)_{A,B}$ or $(\phi + \pi/2, \alpha)_{C,D}$ equal to
FIG. 8. (Color online) Chiral superconducting behavior can be observed in the absence of magnetic field in the rhombus JJ circuit with TBSs at the spinal sites. In (a) and (b), the superconducting phases $\phi$ and $\alpha$, respectively, are shown as functions of the couplings ratios $(J_1/J_{12})$ and $(J_2/J_{12})$ for negative $J_{12}$. The region of chirality corresponds to finite interband current as shown in (c). In (d), the respective energy of the rhombus circuit is shown for $J_{12} = -1$. The two possible signs in Eq. (26) reflect the symmetries stated above. These chiral solutions exist only if $|\gamma^\pm| \leq 1$. However, we emphasize that these nonchiral solutions of the triangular JJ circuit in the absence of magnetic flux correspond to chiral states of the rhombus JJ circuit due to presence of magnetic field ($\alpha \neq 0$), that is, states with finite persistent currents.

In Fig. 8, the phases $\phi$ and $\alpha$ are displayed for the case of zero magnetic flux and TBSs in a sign-reversed superconducting state ($J_{12} < 0$). Regions of chirality ($\phi \neq 0$ or $\alpha \neq 0$) are observed that lead to persistent currents in the JJ array and in particular to a finite interband current [see Fig. 8(c)]. The respective energy of the rhombus is shown in Fig. 8(d). Note that the energy is linear in the couplings ratios $(J_1/J_{12})$ and $(J_2/J_{12})$ if persistent currents are absent. Second-order transition curves separate the chirality regions from the nonchiral ones.

Rather simple solutions are obtained when $J_1 = J_2 = -J_{12} = 1$. In that case, $u = v = -2 \cos \alpha$ for $0 < \Phi_f < \pi$.

The $0, (\pi, 0), (0, \pi)$, or $(\pi, \pi)$ and chiral solutions given by

$$\phi = \pm \cos^{-1}(y^-) - \delta(\Phi_f),$$

$$\alpha = \phi \pm \text{sgn}(\frac{u}{v}) \cos^{-1}(y^+) + \delta(\Phi_f),$$

where

$$y^\pm = \frac{\pm u^2 \mp v^2 - u^2 v^2}{2uvw^\pm},$$

with $w^+ = v, w^- = u$, and

$$u = \frac{J_{10}}{J_{12}}, \quad v = \frac{J_{20}}{J_{12}}, \quad \delta(\Phi_f) = 0, \quad \text{if} \quad 0 < \Phi_f < \pi,$$

$$u = \frac{J_{11}}{J_{12}}, \quad v = \frac{J_{21}}{J_{12}}, \quad \delta(\Phi_f) = \frac{\pi}{2}, \quad \text{if} \quad \pi < \Phi_f < 2\pi.$$
\begin{equation}
J = J_1 + J_2
\end{equation}

and $u = v = -2 \sin a$ for $\pi < \Phi_f < 2\pi$. This leads to $\gamma^- = \gamma^+ = \cos(a)$ and $\gamma^- = \gamma^+ = \sin(a)$, respectively. Therefore one of the solutions would be $\phi = a$ and $\alpha = 2a$, for $0 < \Phi_f < \pi$ and $\phi = -a$ and $\alpha = -2a + \pi$, for $\pi < \Phi_f < 2\pi$ [see Fig. 9(b)]. As before, other solutions can be obtained by symmetry transformations.

In Fig. 9(a), we show the phase diagram of the rhombus in the $(J_1/J_{12})-(J_2/J_{12})$ plane. In the absence of magnetic flux, the region of chirality is shown in red (dark region). Second-order phase transitions to nonchiral states occur at the boundaries of this region. If magnetic flux is applied, finite persistent currents appear in the rhombus circuit, the currents are zero. The existence of finite magnetic flux threading the rhombus only renormalizes the JJ couplings of the triangular circuit and a region of chirality with the same shape is still present in the phase diagram. This region of chirality in the triangular circuit reflects the interdependence of the currents in the rhombus circuit, that is, for $\phi = 0$ or $\pi$ and $\alpha = 0$ or $\pi$, all currents in the rhombus are proportional to $\pm \sin(a)$ or zero and apparently independent from each other. As magnetic flux is increased, the chirality region expands and when $\Phi_f = \pi$ it becomes the region delimited by the dashed blue curve. Increasing further the magnetic flux, the chirality region shrinks and the phase diagram becomes the initial one when $\Phi_f = 2\pi$.

In Figs. 9(b) and 9(c), the superconducting phase $\alpha$ and the energy of the rhombus as functions of the magnetic flux are displayed for the points $P_1$, $P_2$, $P_3$, $P_4$ indicated in Fig. 9(a). A transition is observed at $\Phi_f = \pi$ reflecting the jump from equal phases at the edge sites to opposite phases.

\section*{IV. RHOMBI CHAIN WITH TWO-BAND SUPERCONDUCTORS}

The Hamiltonian of a rhombi chain of Josephson junctions with TBSs at the spinal positions is given by

\begin{equation}
H = - \sum_{n=1,2} J_n \left[ \cos \left( \phi_{ni}^i - \phi_i^+ - A_i^n \right) 
+ \cos \left( \phi_{ni}^s - \phi_i^b - A_i^{sd} \right) 
+ \cos \left( \phi_{ni}^a - \phi_i^+ + A_i^{ad} \right) 
+ \cos \left( \phi_{ni}^b - \phi_i^b + A_i^{bd} \right) \right] 
- J_{12} \sum_i \left[ \cos \left( \phi_i^i - \phi_2^2 \right) \right].
\end{equation}

(28)

where $\phi_i^i$, $\phi_i^s$, $\phi_i^b$ are the phases at the spinal position, at the top edge position and at the bottom edge position, respectively,
and the index \(i\) indicates the unit cell of the rhombi chain. The associated Peierls phases \(\Phi_i\) follow the same notation. This energy can interpreted as the energy of two rhombi chains of OBSs (one for each band) and the energy of a periodic interband coupling. This division is equivalent to that performed in the previous section for the case of one rhombus.

The behavior of the rhombi chain with TBs at the spinal positions under magnetic field shows surprising features near full frustration, \(\Phi_i = \pi\). Below, we discuss the behavior of the rhombi chain, addressing first the zero flux case, second, the case when the rhombus plaquettes are threaded by a finite magnetic flux \(\Phi_f\) such that \(-\gamma \pi < \Phi_f < \gamma \pi\) (where \(\gamma\) is a fraction of one which depends on the number of rhombi in the chain and on the Josephson couplings, and that goes to 1 as the number of rhombi grows), and finally, the near full frustration case where \(\gamma \pi < \Phi_f < (2 - \gamma)\pi\). The behavior of the rhombi chain is periodic in \(\Phi_f\) with period \(2\pi\).

In the absence of magnetic field threading the plaquettes, \(\Phi_f = 0\), the phase configuration associated with the minimum energy of the rhombi chain can be constructed from the minimum energy solutions of one rhombus which have been described in the previous section. However, one must take into account that the rhombi chain has only one interband coupling per unit cell while in the case of the rhombus, two interband couplings contribute to the total energy. Since these two interband energy terms contribute equally to these solutions energy, the solutions of the rhombus with a given interband coupling will allow the construction of the minimum energy phase configuration of the rhombi chain with twice as large interband coupling. So we conclude that the phase diagram of the solutions with an inversion of the currents in a rhombus. Since these two solutions with an inversion of the currents in a rhombus.

If each plaquette is threaded by a finite magnetic flux \(\Phi_f\) such that \(-\gamma \pi < \Phi_f < \gamma \pi\), the same reasoning can be applied and again the evolution with flux of the chiral region in the phase diagram shown in Fig. 9 applies to the rhombi chain with the same renormalization of the axes. In this case, the energy is \(N_f\) times the energy of the rhombus (with \(J_{12} \rightarrow J_{12}/2\) to account for only one interband coupling per plaquette),

\[
E^4 = -2N_r J_{1\delta} \cos \left( \frac{\Delta \Psi_i + \delta}{2} \right) - 2N_r J_{2\delta} \cos \left( \frac{\Delta \Psi_2 + \delta}{2} \right) - N_r J_{1\delta} \cos(\alpha),
\]

where \(\Delta \Psi_i\) and \(J_{1\delta}\) are given by

\[
\Delta \Psi_i = \left( \frac{\Phi_i}{N_f} + 2\phi \right),
\]

\[
\Delta \Psi_2 = \left( \frac{\Phi_i}{N_f} + 2\phi - 2\alpha \right),
\]

\[
J_{1\delta} = 2J_i \left| \cos \left( \frac{\Phi_f - \delta}{4} \right) \right|.
\]

and where \(\delta = 0\) or \(\pi\), whichever gives the minimum energy.

The previous energy can be interpreted as the one resulting from a JJ two-leg ladder (with a vertical coupling \(J_{\delta}\)) at every other site and horizontal couplings \(J_{1\delta}\) and \(J_{2\delta}\) in the top and bottom leg, respectively) with periodic boundary conditions and zero flux threading the plaquettes but a finite inner flux \(\Phi_i\) in the ring. As expressed above, these couplings are dependent on the magnetic flux \(\Phi_f\).

If a multiple of \(2\pi\) is added to the flux \(\Phi_i\), the energy does not change, reflecting a \(2\pi\) periodicity of the energy-\(\Phi_i\) plots as in the case of the one-band rhombus chain, and the superconducting phases gain a plane wave extra phase as explained for the case of the linear chain.

Let us now consider the rhombi chain near full frustration, \(\gamma \pi < \Phi_f < (2 - \gamma)\pi\). If Eq. (29) remained valid as the flux \(\Phi_f\) approached \(\pi\), one would observe similar behavior to that described for the one-band chain, that is, halving of the periodicity of the parabolic-like profile of the energy as a function of the inner magnetic flux \(\Phi_i\) due the energy lowering of the solutions with an inversion of the currents in a rhombus. The halving of the periodicity as the flux \(\Phi_f\) approaches \(\pi\) is indeed observed but with new structure in the energy profile as shown in Figs. 10(a) and 10(c).

In Fig. 10(a), we show \(E(\Phi_i)\) plots for the following values of the magnetic flux through the plaquettes: \(\Phi_f = 0\), \(\Phi_f = 0.9\pi\) or \(1.1\pi\); \(\Phi_f = 0.996\pi\) or \(1.004\pi\); \(\Phi_f = \pi\). The respective Josephson couplings are \(J_1 = 0.6\), \(J_2 = 0.55\) and the interband coupling is \(J_{12} = -2\), and this set of couplings corresponds to a point within the chiral phase in Fig. 9(a). One observes that as \(\Phi_f\) approaches \(\pi\), besides the appearance of the solution associated with the inversion of the current in one rhombus (green dashed curve), new solutions appear (red thick curves) which expand as \(\Phi_f\) approaches \(\pi\) and for \(\Phi_f = \pi\), these are the only solutions present. These new phase configurations do not appear when the set of couplings corresponds to a nonchiral point in the phase diagram in Fig. 9(a) and they are characterized by the breaking of the translation symmetry in the energy density of the JJ array state. In fact, while for \(-\gamma \pi < \Phi_f < \gamma \pi\), the superconducting phases configurations are such that each plaquette carries the same energy, in contrast, for \(\gamma \pi < \Phi_f < (2 - \gamma)\pi\), these new phase configurations imply a modulated energy density and in particular, for \(\Phi_f = \pi\), a doubling of the periodicity of the energy density is observed.

In Fig. 10(c), we show \(E(\Phi_i)\) plots when \(\Phi_f = \pi\), \(J_{12} = -2\), and for several sets of Josephson couplings values [which correspond to points along the diagonal of the phase diagram of Fig. 9(a)]: \(J_1 = J_2 = 0.9\); \(J_1 = J_2 = 1.1\); \(J_1 = J_2 = 1.2\); \(J_1 = J_2 = 1.3\). One observes in the top plot of Fig. 10(c) that the energy profile is completely flat. This behavior is...
FIG. 10. (Color online) In (a), $E(\Phi_1)$ plots are shown for the following sequence of magnetic flux values through the plaquettes: $\Phi_f = 0$; $\Phi_f = 0.9\pi$ or $1.1\pi$; $\Phi_f = 0.996\pi$ or $1.004\pi$; $\Phi_f = \pi$. The set of couplings $J_1 = 0.6, J_2 = 0.55$ and $J_{12} = -2$ corresponds to a point within the chiral phase in Fig. 9(a). In (c), $E(\Phi_1)$ plots are displayed when $\Phi_f = \pi$, $J_{12} = -2$, and for several values of Josephson couplings [corresponding to points along the diagonal of the phase diagram of Fig. 9(a)]: $J_1 = J_2 = 0.9$; $J_1 = J_2 = 1.1$; $J_1 = J_2 = 1.2$; $J_1 = J_2 = 1.3$.

In (b) and (d), we show the supercurrent plots corresponding to the energy plots shown in (a) and (c). (e) Example of a current configuration in the rhombi chain with two-band spinal elements, where a doubling of periodicity is observed. Parameters: $J_1 = 0.6, J_2 = 0.6, J_{12} = -2$, $\Phi_f = \pi$, and $\Phi_i = 0$.

observed as long as $2J_1/J_{12} = 2J_2/J_{12} < 1$. Small shifts from the diagonal of the phase diagram of Fig. 9(a) recover the parabolic-like profile but with a small amplitude of the $E(\Phi_1)$ oscillations as one can observe in the bottom plot of Fig. 10(a). In the second and third plots of Fig. 10(c), plateaus intercalating sinusoidal curves are observed in the energy profiles. These plateaus are observed for $2J_1/J_{12} = 2J_2/J_{12} \gtrsim 1$, with $\{J_1, J_2, J_{12}\}$ corresponding to a point within the chiral region of Fig. 9(a). If the Josephson couplings are increased further, the plateaus disappear and one observes the usual parabolic-like oscillations. If $\Phi_f$ deviates from $\pi$, the plateaus remain almost flat but become narrower following the behavior observed in Fig. 10(a).

When the energy becomes independent of the inner flux in the rhombi chain, and consequently, plateaus are observed in the energy-phase plots, the respective supercurrent along the chain is therefore blocked and the usual sawtoothlike supercurrent-phase plot of the rhombi chain becomes intercalated with zero current regions. This is observed in Figs. 10(b) and 10(d), which show the supercurrent plots corresponding
to the energy plots of Figs. 10(a), and 10(c), respectively. We emphasize that this blocking of the supercurrent along the chain is much stronger than the blocking of supercurrent in the one-band rhombus chain, which occurs only at full frustration.

In Fig. 10(e), we present an example of a current configuration with broken translation symmetry in the rhombi chain with two-band spinal elements. The parameters are the same as those of the first plot of Fig. 10(c) \((J_1 = 0.6, J_2 = 0.6, J_{12} = -2, \Phi_f = \pi \text{ and } \Phi_i = 0)\). A clear doubling of periodicity of the current configuration is observed (compare it with Fig. 4(b)), with every other TBS displaying zero interband supercurrent, that is, having a \(\pi\) interband phase difference (an exactly sign-reversed state). These interband \(\pi\) junctions block the supercurrent along the chain when \(J_1 = J_2\), since in this case the sum of Josephson energies between this TBS and an OBS to its right (or left) is zero, for any value of the phase \(\alpha\) [see the labeling of phases in Fig. 10(e)] of the first band of the TBS. This implies that this sign-reversed TBS effectively decouples from the rest of the chain. In terms of Josephson currents, this implies that the supercurrent flowing in the junction between the sign-reversed TBS and a OBS is zero. However, the current from each of the bands of the TBS to the OBS is not zero, despite the fact the interband current is zero. As one may observe in Fig. 10(e), the current flows from the sign-reversed TBS to the OBSs on its right (or left) drawing a closed circuit: from band 1 of the sign-reversed TBS to the top OBS, then to band 2 of the TBS, then to the bottom OBS and finally closes the circuit returning to band 1. The same circuit with current flowing in the opposite direction also occurs.

The phases \(\theta_i, i = 1, 2, 3, 4\), and \(\Delta\), shown in Fig. 10(e) can be determined when \(J_1 = J_2\) and \(\Phi_f = \pi\) since the respective region of the chain [region delimited by the dashed rectangle in Fig. 10(e)] is decoupled from the rest of the chain due to the sign-reversed TBS on its left and right. Minimizing the Josephson energies terms in this region that contain the phase \(\theta_i\), one obtains \(\theta_1 = \Delta - \pi/4 - \Phi_f/(2N_r)\) for negative \(J_1/J_{12}\). Doing the same for the other phases, one obtains \(\theta_1 = \theta_2 = \Delta - \pi/4 - \Phi_f/(2N_r)\) and \(\theta_3 = \theta_4 = \Delta + \pi/4 - \Phi_f/(2N_r)\). So the energy associated with the decoupled region is \(-J_{12} \cos(2\Delta) - 8J_1 \cos \Delta, \text{ independent of } \Phi_i\) and, therefore, we will see a plateau in \(E(\Phi_f)\) as long as this phase configuration is the lowest energy one. This independency of \(\Phi_f\) is consistent with the fact that the decoupled region is an open circuit. Minimizing the previous energy expression, one obtains \(\cos \Delta = -2J_1/J_{12}\) (the two solutions for \(\Delta\) correspond to opposite directions of the currents flowing in the decoupled region) and therefore, the energy of the decoupled region is \(E_{\text{flat}} = 4N_rJ_1^2/J_{12}\) and, since the energy of the sign-reversed TBS is \(J_2\), the total energy of the chain is

\[
E_{\text{flat}} = N_rJ_{12} + 4N_rJ_1^2/J_{12}
\]

(31)

(we assume \(N_r\) even). This expression agrees exactly, for example, with the value \(-21.72\) given the parameters of the top plot of Fig. 10(c).

If \(J_1\) is different from \(J_2\) [see Fig. 10(a)], the solution (red thick curve) with modulated energy density has the same form, with every other TBS in an exactly sign-reversed state, but now current flows along the chain since the current from band 1 of the TBS to a OBS is not exactly compensated by the current from the same OBS to band 2 of the TBS. This is reflected by the small but finite \(\Phi_f\) dependence of the total energy in the bottom plots of Fig. 10(a). In these two plots, the two different solutions corresponding to the paraboliclike arcs of the red thick curve differ in the currents direction in a short region of the chain, which is similar to what we have described in Sec. II B (in the case of the rhombi chain of OBS, the lowest energy solutions differ in the current direction in one rhombus).

The existence of these energy plateaux (and the consequent blocking of supercurrent) can also be qualitatively understood establishing a parallel with the existence of flat bands in the corresponding geometrically frustrated tight-binding models. In Appendix A, we explain how to construct a close analogy between such tight-binding models and JJ arrays. This analogy is not exact when the tight-binding ground states are not homogeneous, but it still allows us to explain qualitatively the unusual behavior of the JJ rhombi chain. If the tight-binding ground states fall into a flat band as the magnetic flux is varied, they will be localized states and the respective energy will be independent of the magnetic flux \([23]\). This leads to a plateau in the ground-state energy plot just as we have observed in the JJ rhombi array. Due to the high degeneracy of a flat band, the numerically obtained phase configurations for the JJ rhombi will correspond to a superposition of the localized tight-binding states. The supercurrent blocking is a consequence of the energy being independent of magnetic flux in this case.

In Fig. 11, we show the amplitude of the \(E(\Phi_f)\) oscillations as a function of \(\Phi_f\) \((E_{\text{max}} - E_{\text{min}}\) on the left axis\), and minimum value of the energy as a function of \(\Phi_f\) \((E_{\text{min}}\) on the right axis\) of the rhombi chain with TBSs at the spinal sites. The amplitude decreases much more [compared with Fig. 4(d)] at \(\Phi_f = \pi\) due to the existence of new solutions, which break the translation symmetry of the energy distribution along the rhombi chain. Parameters: \(J_1 = 0.6, J_2 = 0.55, J_{12} = -2\).

FIG. 11. (Color online) Amplitude of the energy-phase oscillations as a function of \(\Phi_f\) \((E_{\text{max}} - E_{\text{min}}\) on the left axis\), and minimum value of the energy as a function of \(\Phi_f\) \((E_{\text{min}}\) on the right axis\) of the rhombi chain with TBSs at the spinal sites. The amplitude decreases much more [compared with Fig. 4(d)] at \(\Phi_f = \pi\) due to the existence of new solutions, which break the translation symmetry of the energy distribution along the rhombi chain. Parameters: \(J_1 = 0.6, J_2 = 0.55, J_{12} = -2\).

V. CONCLUSION

In this paper, we have described modifications to the behavior of a single rhombus and of a rhombi array of Josephson junctions in the large capacitance limit due to
presence of sign-reversed TBSs at the spinal positions of the rhombi chain. We have shown that these JJ circuits can be mapped onto a triangular JJ circuit and a JJ two-leg ladder, respectively, with zero effective magnetic flux, but with effective Josephson couplings which are magnetic flux dependent. This mapping leads to an additional chirality variable, which is different from the chirality induced by magnetic field in the rhombus plaquette. In fact, one may have finite currents in the rhombi chain and no current in the mapped JJ array. A phase diagram of this mapped chirality was constructed for both the rhombus and the rhombi chain in the plane of the Josephson couplings. The effect of a frustrating magnetic flux is the expansion/contraction of the mapped chiral regions and consequently, transitions between chiral and nonchiral mapped states may occur. Increasing temperature leads to variation of the ratios between Josephson couplings as well as between the Josephson couplings and the interband couplings, and transitions between chiral and nonchiral states may also be observed.

We have also shown that the supercurrent along the rhombi chain exhibits rather unusual behavior, when the mapped state of the rhombi chain falls into the chiral region of the phase diagram. Near full frustration and for equal Josephson couplings, plateaus are observed in the energy-phase plots and the respective supercurrent along the chain is blocked. The usual sawtoothlike supercurrent-phase plot of the rhombi chain is therefore intercalated with zero current regions that become wider as the Josephson couplings are decreased. The blocking of the supercurrent along the chain is due to the locking of every other TBS in an exactly sign-reversed state (which implies a doubling of the periodicity of the energy density along the chain).

The existence of chiral states in the mapped arrays reflect the frustration induced by the sign-reversed two-band superconducting elements and therefore may be used has a probe of the relative phase of the superconducting gaps in TBSs. The classical behavior of the one-band rhombi chain has been experimentally observed [5,6,22] and we suggest that the results presented in this paper should also be experimentally observed in a rhombi chain with pnictide superconductors at the spinal positions if indeed a sign-reversed superconducting state occurs in these systems [20].

If the charging energy is relevant compared with the Josephson energy of the junctions, quantum modifications should occur in the classical behavior of the rhombi chain with two-band superconducting elements and therefore may be used as a probe of the relative phase of the superconducting gaps in TBSs. The classical behavior of the one-band rhombi chain has been experimentally observed [5,6,22] and we suggest that the results presented in this paper should also be experimentally observed in a rhombi chain with pnictide superconductors at the spinal positions if indeed a sign-reversed superconducting state occurs in these systems [20].

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APPENDIX A: JJ ARRAYS IN THE HIGH-CAPACITANCE LIMIT

In this Appendix, we first introduce the high-capacitance approximation for the Hamiltonian of a JJ array. Next, we consider a JJ array with two-band superconducting elements and show that the two bands of these superconductors play a similar role to two OBSs. Finally, we discuss the limits of weak and strong interband coupling in these arrays with TBSs.

A JJ is created when two superconducting islands are connected by a constricted superconducting region or by a thin layer of a metallic or insulating material. The Josephson coupling of the superconducting order parameters induced by this link depends on its characteristics, for example, the thickness of the insulating barrier, and one may consider it a controllable parameter. In the weak-link approximation, the Josephson tunneling is a small interaction which affects only the relative phases of the superconducting order parameters in each island but not the magnitude of the superconducting gaps. A finite phase difference between the superconducting islands leads in general to the appearance of a Josephson current [25].

The JJ Hamiltonian must consider not only the Josephson energy associated with the tunneling of Cooper pairs but also the charging energy due to the electrostatic interaction within each superconducting island. The charging energy is inversely proportional to the capacitance of the island and for sufficiently large islands, this energy contribution can be neglected. The Hamiltonian that describes arrays of Josephson junctions in this high-capacitance limit is

$$H = - \sum_{\langle ij \rangle} J_{ij} \cos(\phi_i - \phi_j - A_{ij}),$$

(A1)

where \(\phi_i\) is the superconducting phase of the island \(i\) and \(A_{ij}\) is the phase shift due to the presence of an external magnetic field, obtained from the line integral of the vector potential along the path from site \(i\) to site \(j\),

$$A_{ij} = \frac{2\pi}{\Phi_0} \oint_{ij} \mathbf{A} \cdot d\mathbf{l},$$

(A2)

where \(\mathbf{A}\) is the vector potential, corresponding to a uniform transverse magnetic field \(B\) and \(\Phi_0\) is the superconducting flux quantum, \(\Phi_0 = h/(2e)\). Considering a 2D JJ array, the uniform frustration \(f\) is then related to \(B\) by the relation \(f = BA_p/\Phi_0\), where \(A_p\) is the plaquette area. A fully frustrated JJ array is obtained with a half-flux quantum per plaquette. This magnetic flux is equivalent to the change of the sign of one of the Josephson couplings from positive to negative, that is, it creates a \(\pi\) junction [19].

There is a close relation between fully frustrated JJ arrays, itinerant geometrically frustrated electronic systems and magnetic frustrated systems. JJ arrays in the high-capacitance limit are physical realizations of the classical XY model where frustration occurs in the case of an odd number of antiferromagnetic couplings per plaquette [9–11].
The Hamiltonian of a JJ array can also be mapped onto a one-particle tight-binding model in a lattice with the same geometry and under the same magnetic field [10,26]. In certain geometries, this mapping of the JJ array Hamiltonian into a tight-binding problem simplifies considerably the analysis. The Hamiltonian given by Eq. (A1) can be written in the compact form \( H = -\frac{1}{2} \mathbf{v}^\dagger \mathbf{P} \mathbf{v} \), where \( \mathbf{P} \) is a Hermitian and positive definite matrix with elements \( P_{ij} = J_j e^{-i k_i} \) and \( \mathbf{v} \) is a column vector with components \( v_i = e^{-i \phi_i} \). The \( \mathbf{P} \) matrix can be interpreted as the Hamiltonian matrix of a one-particle tight-binding model and the diagonalization of the \( \mathbf{P} \) operator provides directly the minimum energy configuration of the JJ array if the ground state of the tight-binding model has uniform density.

1. JJ arrays with two-band superconducting elements

The Hamiltonian defined in Eq. (A1) implicitly assumes that the respective Josephson energy is much smaller than the condensation energies of the superconductors in the JJ array and therefore the value of the gap function of each of the OBSs is determined from the usual BCS gap equation without taking into account the energy contribution of Eq. (A1). In the case of a TBS, the interband tunneling energy term may be comparable to the intraband condensation energy and the gap functions are determined taking also into account this energy contribution [27]. The temperature dependence of the gap functions in the case of TBSs with interband pair-tunnelling is long known [28]. The usual starting point for the determination of the gap functions is the reduced BCS Hamiltonian generalized to two bands [28].

\[
H = \sum_{i,k} \varepsilon_i c_{i,k}^\dagger c_{i,k} - \sum_{i,j,k,k'} V_{ij} c_{i,k}^\dagger c_{j,-k,k'} c_{j,-k',k} c_{j,k'}^\dagger ,
\]

(A3)

with \( \varepsilon_i = \varepsilon_i - \mu \) and where \( i, V, \) and \( \mu \) are, respectively, the band index (\( i = 1, 2 \)), the interaction constant, and the chemical potential. The \( k \) sums in the interaction term follow the usual BCS restrictions.

Applying a mean-field approach to this Hamiltonian, minimization equations are obtained for the superconducting phases and the superconducting gaps. Below, we present the approach followed in Ref. [15] in order to determine these equations in the particular case of a TBS.

At zero temperature, considering the usual BCS operators \( \Psi_i = \sum_k c_{i,k}^\dagger c_{i,-k,k} \) in the subspace constructed from the set of BCS states [27], written as

\[
| \mathcal{F}(\Delta, \Phi) \rangle = \prod_{j=1}^{2} \left[ \prod_k (u_{jk} + e^{i \phi_j} v_{jk} c_{j,k}^\dagger c_{j,-k,k}^\dagger) \right] | \Phi_0 \rangle ,
\]

(A4)

one obtains

\[
\hat{\Psi}_j | \mathcal{F}(\Delta, \Phi) \rangle = \Psi_j | \mathcal{F}(\Delta, \Phi) \rangle = \sum_k u_{jk} v_{jk} e^{-i \phi_j} | \mathcal{F}(\Delta, \Phi) \rangle ,
\]

(A5)

where \( \Delta = (\Delta_1, \Delta_2) \) and \( \Phi = (\phi_1, \phi_2) \) are the absolute values of the superconducting gaps and the respective phases, with the usual BCS definition of superconducting order parameters

\[
u_{ik} v_{ik} = \frac{\Delta_i}{2 E_{ik}},
\]

(A6)

where \( E_{ik} = \sqrt{\varepsilon_i^2 + \varepsilon_k^2} \) and \( u_{ik}^2 + v_{ik}^2 = 1 \) (\( u_{ik} \) and \( v_{ik} \) are real). The Hamiltonian is also diagonal in the BCS basis and its eigenvalues are given by

\[
E = \sum_i f_i (| \Psi_i |^2) - \sum_i V_{ii} | \Psi_i |^2 - \sum_{i \neq j} V_{ij} | \Psi_i | | \Psi_j | \cos(\phi_j - \phi_i),
\]

(A7)

where \( f_i (| \Psi_i |^2) \) is the kinetic energy contribution of the respective band term in the Hamiltonian and has an elaborate dependence on the superconducting parameters \( \Delta_i \). The other terms result from the intraband and interband pairing terms in the Hamiltonian. At finite temperature, this expression remains valid with the modification [15]

\[
\Psi_j = \sum_k u_{jk} v_{jk} e^{-i \phi_j} (1 - 2 f_{jk}),
\]

(A8)

where \( f_{jk} \) is the occupation number of the quasiparticle state with momentum \( k \) and energy \( E_{jk} \) of the band \( j \). The minimization of the free energy with respect to the phases \( \phi_i \) gives

\[
\frac{\partial E}{\partial \phi_i} = 0, \quad \forall i \Rightarrow \sum_j V_{ij} | \Psi_j | \sin(\phi_j - \phi_i) = 0, \forall i.
\]

(A9)

The minimization of the free energy with respect to the absolute values of the superconducting parameters [27,29] leads to the system of coupled gap equations

\[
\Delta_{jk} = \sum_k V_{kk}^{ij} \cos(\phi_j - \phi_i) u_{jk} v_{jk} (1 - 2 f_{jk}),
\]

(A10)

with \( j = 1, \ldots, n \), which, following the usual steps [29], can be written as

\[
\Delta_i = \sum_j V_{ij} \cos(\phi_j - \phi_i) \int_0^{\omega_D} d \xi K_j(\xi, \Delta_j, T) \Delta_j ,
\]

(A11)

with

\[
K_j(\xi, \Delta, T) = \frac{N_j(\xi)}{E \ tanh \frac{\beta E}{2}},
\]

(A12)

where \( E = \sqrt{\xi^2 + \Delta^2} \), \( \omega_D \) is the usual frequency cutoff, \( N_j(\xi) \) is the density of states of the \( j \) band and \( 1/\beta = k_B T \).

In the case of a TBS, the minimization in order to the superconducting phases leads to a zero phase difference if the interband coupling is attractive or \( \pi \) if it is repulsive. In both cases, the same temperature dependence for the absolute values of the gap functions is obtained. This temperature dependence is displayed in Fig. 12 for increasing values of the interband coupling. For small interband coupling, the gap functions grow and a clear change of the behavior of the lower superconducting gap near the critical temperature occurs.
A single critical temperature is present even in the limit of very small interband tunneling. Besides a global enhancement of the superconducting gaps with increasing interband coupling, one can observe a clear change of the behavior of the lower superconducting gap near the critical temperature.

The energy of an array of JJs with one-band and TBSs can be described by a similar expression to that of Eq. (A7), if we add the BCS energy to the Josephson term given by Eq. (A1) in the case of the OBSs. The total energy becomes

$$E = \sum_i f_i(|\Psi_i|^2) - \sum_i V_{ii} |\Psi_i|^2 - \sum_{ij} J_{ij} \cos(\phi_i - \phi_j - A_{ij}).$$

(A13)

The previous expression describes the energy of a JJ array with one band and TBSs, if the index $i$ labels not only the one-band superconducting elements but also each of the bands of the TBSs. This energy must be minimized with respect to the absolute value of the superconducting gaps as well as with respect to the superconducting phases.

2. Limiting cases

Two limiting situations may be considered in what concerns the TBSs: energy contribution due to interband pairing of the order of the Josephson energy between different superconductors or much larger than this Josephson energy.

If the energy contribution due to interband pairing is of the order of the Josephson energy, and since the weak limit is considered for the JJs, this implies that this energy is negligable compared with the intraband contribution to the condensation energy of the TBSs. Therefore the superconducting gaps of the TBSs can be determined independently (using the one-band gap equation) for temperatures lower than the lowest of the critical temperatures of the independent bands. For larger temperatures, the interband coupling is relevant for the band with lower condensation energy inducing a finite but small gap in the band. However, this contribution remains irrelevant for the band with higher condensation energy. The relation between the two gaps in this temperature range is particularly simple, $\Delta_2(T) = V_{12}/V_{11} \Delta_1(T)$ due to the interband contribution in the calculation of the superconducting gap for the second band $\Delta_2(T)$. This expression should be modified due to the influence of the Josephson tunnellings between superconductors, but not significantly if $\Delta_1(T)$ is the largest superconducting gap of the JJ array in this temperature range.

If the energy contribution of the interband coupling in the TBSs is much larger than the Josephson energy between different superconductors, it has to be considered in the determination of the gap values of the TBS. The weak-link limit is assumed for the Josephson tunnellings between different superconductors and therefore the Josephson energies are neglected in the determination of the superconducting gaps. This case is characterized by a reduction on the number of independent tunnellings, which can be interpreted as an effective modification of the geometry of the JJ circuit. Consider, for example, a square JJ circuit of OBSs where one of the JJ couplings (let us assume $J_2$) is much larger than the others. This implies $\phi_3 \approx \phi_2$ and $\phi_1 \approx \phi_2 + \pi$ if $J_2$ is positive or negative, respectively. Note that the energy minimization (in order to $\phi_2$) implies $J_1 \sin(\phi_1 - \phi_2) \approx J_2 \Delta \phi_{23}$, where the sine has been expanded on the right term of the equation and $\Delta \phi_{23} = \phi_2 - \phi_1$ or $\Delta \phi_{23} = \phi_2 - \phi_1 + \pi$ if $J_2$ is positive or negative, respectively. This implies that this phase difference is small, $\Delta \phi_{23} \approx (J_1/J_2) \sin(\phi_1 - \phi_2)$ and this leads to the Josephson current relation:

$$J_1 \sin(\phi_1 - \phi_2) = J_2 \sin(\phi_2 - \phi_1)$$

$$= J_1 \sin(\phi_3 - \phi_4)$$

$$\approx \begin{cases} J_3 \sin(\phi_2 - \phi_4), & J_2 > 0, \\ -J_3 \sin(\phi_2 - \phi_4), & J_2 < 0. \end{cases}$$

(A14)

This system can be interpreted as a triangular JJ circuit with superconducting phases $\phi_1$, $\phi_2$ and $\phi_4$ and with JJ couplings $J_1$, $\pm J_3$ and $J_4$ where the sign associated with $J_3$ is given by the sign of $J_2$.

APPENDIX B: TEMPERATURE DEPENDENCE OF THE JOSEPHSON AND INTERBAND COUPLINGS

Specific to JJ arrays is the temperature evolution of the Josephson couplings which reflects the BCS temperature dependence of the superconducting gaps (which vanish approximately as a square root as the BCS critical temperature is approached). This temperature dependence is absent in magnetically frustrated systems and itinerant geometrically frustrated electronic systems since the Heisenberg couplings and transfer integrals are considered to be temperature independent. This temperature dependence is usually neglected in JJs arrays experiments since (i) these experiments are carried out well below the BCS critical temperature and (ii) all superconducting islands are assumed to be identical so that the only available variables are the superconducting phases of the islands.

Two different tunnellings are considered in this paper: the usual Josephson tunneling between two OBSs and interband tunneling in the case of a single TBS. The minimization of the energy with respect to the superconducting phases leads to

$$\sum_j J_{ij} \sin(\phi_j - \phi_i) = 0.$$  

(B1)
where \(i\) is the “site” of the JJ array (recall that in the case of a TBS, each band is considered a site as shown in Fig. 2) and \(J_{ij} = 2V_{ij}/|\Psi_i||\Psi_j|\) in the case of the interband tunneling.

As mentioned before, the condensation energy corrections due to the interband interactions can be of the order of the intraband contributions and therefore the determination of the temperature dependence of the interband couplings \(J_{ij} = 2V_{ij}/|\Psi_i||\Psi_j|\) requires the solution of the coupled gap equations, Eq. (A11). In these equations, the relative difference of superconducting phases associated with each band of a TBS is also present. However, since the interband coupling is much larger than the Josephson constant for tunneling between different superconductors, this relative phase will be approximately 0 or \(\pi\) for attractive or repulsive interband interaction, respectively. This will be discussed in more detail in the next section. The gap values for the TBS will be determined considering these values for the relative phase and the typical temperature dependence is displayed in Fig. 12 for several values of the interband coupling.

The temperature dependence for the Josephson-like constant in the case of the pair tunneling between the bands \(i\) and \(j\) of a TBS is determined from

\[
J_{ij} = 2V_{ij}/|\Psi_i||\Psi_j|
\]

while the determination of the temperature dependence for the tunneling constant in the case of a Josephson junction between two OBSs is given by \([30,31]\)

\[
J_{ij}(T) = R_n^{-1}\Delta_i(T)\Delta_j(T)\frac{\pi}{2\beta} \times \sum_{l=0, \pm 1, \ldots} \left\{\left[\omega_l^2 + \Delta_i^2(T)\right]\left[\omega_l^2 + \Delta_j^2(T)\right]\right\}^{-1/2},
\]

where \(\omega_l = \pi(2l + 1)/\beta\), \(R_n\) is the normal state resistance and \(\Delta_i(T)\) is the gap energy of superconductor \(i\) at the temperature \(T\). In the latter case, the gap values are only determined by the intraband pairing and one neglects the influence of the Josephson tunneling in the gap values (i.e., one considers the usual weak-link limit). The energy gaps depend on the temperature and this dependence is obtained for the latter from the gap equation of a OBS,

\[
\frac{2}{N(0)V} = \int_{-\Delta_0}^{\Delta_0} d\xi \frac{1 - 2\left[\left(\xi^2 + \Delta_0^2\right)^{1/2}\right]}{\left(\xi^2 + \Delta_0^2\right)^{1/2}},
\]

where \(\omega_D\) is the Debye frequency and \(N(0)\) is the density of states at the Fermi energy. In the case of the TBS, the interband coupling has to be considered (except if it is very small compared with the intraband coupling) and the temperature dependence is obtained solving the system of coupled gap equations, Eq. (A10).

FIG. 13. (Color online) Temperature evolution of (a) gap functions \(\Delta_1\), \(\Delta_2\), and \(\Delta_3\) (neglecting the contribution of the Josephson tunnellings); (b) interband and Josephson couplings; and (c) superconducting phases \(\phi\) and \(\alpha\) of the rhombus circuit with TBSs at the spinal positions and in the absence of magnetic field. In (a), the two-band gaps \(\Delta_1\) and \(\Delta_2\) vanish at the same critical temperature \(T_{c,12}\) which we chose larger than the critical temperature of the OBSs, \(T_{c,5} \approx 0.77T_{c,12}\) (indicated by the vertical dashed line). With increasing temperature, second-order phase transitions are observed in the superconducting phases plot, first from a nonchiral state to a chiral state and second from a chiral state to a nonchiral state. In these plots, the pairing constants for the TBSs are \(V_{12} = .01\) and \(V_{22} = .88\) in units of \(V_{11}\) and for the OBSs \(V_{1} = .95V_{11}\). The Josephson couplings between the two-band and the one-band superconductors are such that \(J_1(0) \approx 0.08J_{12}(0)\) and \(J_2(0) \approx 0.12J_{12}(0)\).

1. Temperature dependence of the chiral state

The influence of increasing temperature in the chiral superconducting state of a three-band superconductor has been discussed by Dias and Marques \([15]\). This discussion is easily extended to the case of the rhombus circuit, where one has interband couplings as well as Josephson couplings. The temperature dependence of the interband and Josephson couplings has been discussed above. One should note that in general, the OBSs and the TBSs have different critical temperatures and as temperature is increased, the JJ circuit will open since one of the superconductors reached its critical temperature. In that case, the Josephson currents are zero and the superconducting phase differences between superconductors are random.

In Fig. 13, we show the temperature evolution of the gap functions, the interband and Josephson couplings, and the
superconducting phases $\phi$ and $\alpha$ of the rhombus circuit in the absence of magnetic field. With increasing temperature, second-order phase transitions are observed in the superconducting phases plot, first from a nonchiral state to a chiral state and second from a chiral state to a nonchiral state. These transitions lead to slope changes in the gap functions [15] as well as in the tunneling couplings. In this case, these slope changes are very small (only noticeable in the smaller gap function) since (i) the couplings $J_1$ and $J_2$ are very small compared with the interband coupling $J_{12}$; (ii) $J_{12}$ is small compared with the intraband couplings; and (iii) $\alpha$ deviates only slightly from $\pi$.