Entanglement Concentration Using Quantum Statistics

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We propose an entanglement concentration scheme which uses only the effects of quantum statistics of indistinguishable particles. This establishes the fact that useful quantum information processing can be accomplished by quantum statistics alone. Because of the basis independence of statistical effects, our protocol requires less knowledge of the initial state than most entanglement concentration schemes. Moreover, no explicit controlled operation is required at any stage.

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Quantum statistics has led to a number of interesting phenomena such as ferromagnetism, superconductivity, and superfluidity. However, its effects have never been used to process information. Recently, we have shown that quantum statistics can lead to an *effective* interaction between internal and external degrees of freedom of particles [1]. This was demonstrated in the context of entanglement transfer from spin to path without an explicit conditional operation between these degrees of freedom. Going even further in our exploration of the effects of quantum statistics in information theory, we present an entanglement concentration protocol [2,3] based on these effects. This establishes the fact that quantum statistics *alone* (without any other explicit interaction between the relevant degrees of freedom of the particles involved) is sufficient for useful quantum information processing.

The best performances in quantum communication and computation processes are normally achieved using a pair of pure maximally entangled systems (particles) shared between distant parties. But the inevitable influence of the environment during the distribution of the entangled pairs reduces the amount of shared entanglement. As entanglement cannot be increased by local operations and classical communication [4], the only option is to concentrate it locally from a larger to a smaller number of pairs [2,3,5–11]. In the case of mixed states we call these protocols *entanglement purification,* while for pure states we refer to them as *entanglement concentration* [3].

In a setting similar to previous entanglement concentration protocols, we consider the action of specific local operations on two pairs of entangled systems, driving them with some probability into one pair of more entangled systems.

However, unlike most of the previous schemes, we require our entangled systems to be composed of n particles (see Fig. 1 for the representation of a pair of these systems). We look at the entanglement in the internal degrees of freedom of the particles, such as the spin in the case of electrons (fermions) or the polarization in the case of photons (bosons), which have isomorphic Hilbert spaces. The initial pure state of each of our two pairs, distributed between two parties Alice (A) and Bob (B), is

$$|\phi\rangle^n \equiv \alpha|\underbrace{\uparrow\uparrow\cdots\uparrow}_{n}\rangle_A|\downarrow\downarrow\cdots\downarrow\rangle_B + \beta|\downarrow\downarrow\cdots\downarrow\rangle_A|\uparrow\uparrow\cdots\uparrow\rangle_B \tag{1}$$

with $|\alpha|^2 + |\beta|^2 = 1$ and where, for instance, we have

$$|\uparrow\uparrow\cdots\uparrow\rangle_A = \hat{a}^+_{A1\uparrow}\hat{a}^+_{A2\uparrow}\cdots\hat{a}^+_{An\uparrow}|0\rangle \equiv |A\tilde{\uparrow}\rangle^n. \tag{2}$$

Here $\hat{a}^+$ are creation operators that ensure the ordering of the multiparticle state, as the usual commutation and anticommutation rules apply for bosons and fermions, respectively. For the sake of compactness, we will rewrite Eq. (1) as

$$|\phi\rangle^n = (\alpha|A\tilde{\uparrow}\rangle^n|B\tilde{\downarrow}\rangle^n + \beta|A\tilde{\downarrow}\rangle^n|B\tilde{\uparrow}\rangle^n), \tag{3}$$

where the tilde over the arrow reminds us of the string of spins (or polarizations).

Let us now label one pair L and the other R (as in left and right, see Fig. 2). Then, our total initial state is

$$|\varphi\rangle^n \equiv |\phi\rangle^n_L \otimes |\phi\rangle^n_R, \tag{4}$$

where $|\phi\rangle^n_L$ is given by Eq. (1) written for pair L, and similarly for R.

Using Eq. (3), we can rewrite the total state (4) as

$$|\varphi\rangle^n = \alpha^2|A\tilde{\uparrow}\rangle^n_L|A\tilde{\uparrow}\rangle^n_R|B\tilde{\downarrow}\rangle^n_L|B\tilde{\downarrow}\rangle^n_R + \beta^2|A\tilde{\downarrow}\rangle^n_L|A\tilde{\downarrow}\rangle^n_R|B\tilde{\uparrow}\rangle^n_L|B\tilde{\uparrow}\rangle^n_R + \alpha\beta(|A\tilde{\uparrow}\rangle^n_L|A\tilde{\downarrow}\rangle^n_R|B\tilde{\downarrow}\rangle^n_L|B\tilde{\uparrow}\rangle^n_R + |A\tilde{\downarrow}\rangle^n_L|A\tilde{\uparrow}\rangle^n_R|B\tilde{\uparrow}\rangle^n_L|B\tilde{\downarrow}\rangle^n_R). \tag{5}$$

Our protocol consists of bringing each of Bob's particles into a 50/50 beam splitter (particles i from the left and the right systems go into beam splitter i, for $i = 1, 2, \ldots, n$) and then making a specific measurement on the output particles

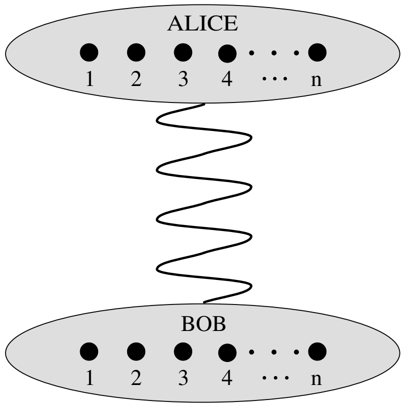

FIG. 1. This figure represents one of the two entangled pairs that we use for our protocol. Each pair is composed of two n-particle states entangled between Alice and Bob with at most 1 e-bit of entanglement.

using detectors Li and Ri (see Fig. 2). Before the particles from the left fall into the beam splitter, we apply to each of them a specific unitary transformation U_π^i, that flips their spin or polarization. This is done in the arm ℓi of each beam splitter and the transformation is defined for every i:

$$U_\pi^i : \begin{cases} |B\uparrow\rangle_{\ell i} \to |B\downarrow\rangle_{\ell i}\,, \\ |B\downarrow\rangle_{\ell i} \to |B\uparrow\rangle_{\ell i}\,. \end{cases} \tag{6}$$

This flipping is applied in order to align some spins (or polarizations) in appropriate terms so that we can exploit the statistical effects later on.

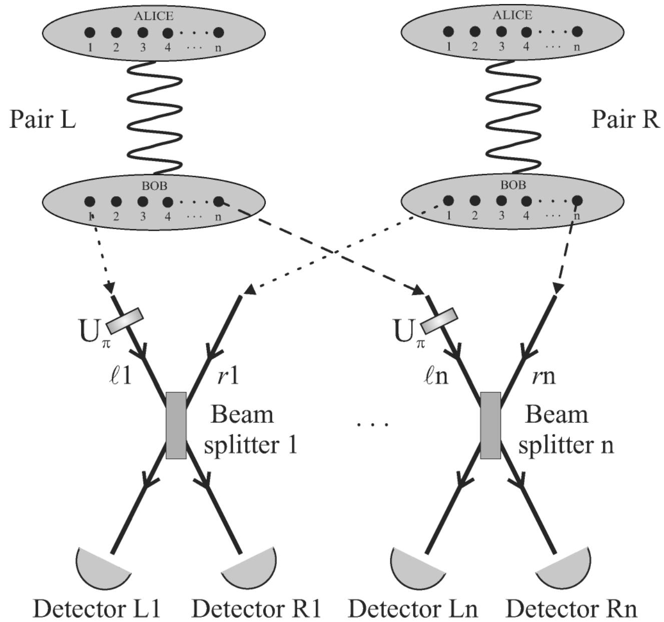

FIG. 2. This figure represents the setup for our entanglement concentration protocol using quantum statistics. Initially, Alice and Bob share two pairs of entangled systems, L and R. Each entangled pair is composed of two n-particle states. One of the parties, say Bob, can then convert these pairs into a more entangled pair by performing a set of local operations using only standard 50/50 beam splitters, path measurements, and one-way classical communication with Alice.

The total state then becomes

$$\begin{aligned} |\varphi_0\rangle^n &\equiv (U_\pi^1 \otimes U_\pi^2 \otimes \cdots \otimes U_\pi^n)|\varphi\rangle^n \\ &= \alpha^2 |A\tilde{\uparrow}\rangle_L^n |A\tilde{\uparrow}\rangle_R^n |B\tilde{\uparrow}\rangle_L^n |B\tilde{\downarrow}\rangle_R^n + \beta^2 |A\tilde{\downarrow}\rangle_L^n |A\tilde{\downarrow}\rangle_R^n |B\tilde{\downarrow}\rangle_L^n |B\tilde{\uparrow}\rangle_R^n + \alpha\beta(|A\tilde{\uparrow}\rangle_L^n |A\tilde{\downarrow}\rangle_R^n |B\tilde{\uparrow}\rangle_L^n |B\tilde{\uparrow}\rangle_R^n + |A\tilde{\downarrow}\rangle_L^n |A\tilde{\uparrow}\rangle_R^n |B\tilde{\downarrow}\rangle_L^n |B\tilde{\downarrow}\rangle_R^n)\,. \end{aligned} \tag{7}$$

First, we present the entanglement concentration protocol for fermions (e.g., electrons) and then the counterpart protocol for bosons (e.g., photons).

Fermions.—After the particles pass through the beam splitter, Bob performs a path measurement on the first pair using detectors $L1$ and $R1$ (see Fig. 2). For simplicity of presentation, we first assume that these detectors do not absorb the particles and do not disturb their internal degrees of freedom (such detectors are indeed available for electrons and atoms and can have interesting applications in deterministically entangling independent particles using indistinguishability and feedback [12]). However, as we will show later on, this is not a necessary requirement for success of our protocol. If the particles (which we assumed to be fermions) bunch, we discard them. Otherwise, in the case of antibunching, we get the pure state

$$\begin{aligned} |\varphi_1\rangle^n = N_1\Bigg[&\frac{1}{\sqrt{2}}(\alpha^2 |A\tilde{\uparrow}\rangle_L^n |A\tilde{\uparrow}\rangle_R^n |B\,\text{triplet}\rangle_{L1R1} |B\tilde{\uparrow}\rangle_L^{n-1} |B\tilde{\downarrow}\rangle_R^{n-1} + \beta^2 |A\tilde{\downarrow}\rangle_L^n |A\tilde{\downarrow}\rangle_R^n |B\,\text{triplet}\rangle_{L1R1} |B\tilde{\downarrow}\rangle_L^{n-1} |B\tilde{\uparrow}\rangle_R^{n-1}) \\ &+ \alpha\beta(|A\tilde{\uparrow}\rangle_L^n |A\tilde{\downarrow}\rangle_R^n |B\tilde{\uparrow}\rangle_L^n |B\tilde{\uparrow}\rangle_R^n + |A\tilde{\downarrow}\rangle_L^n |A\tilde{\uparrow}\rangle_R^n |B\tilde{\downarrow}\rangle_L^n |B\tilde{\downarrow}\rangle_R^n)\Bigg], \end{aligned} \tag{8}$$

where

$$N_1 = \left[\frac{|\alpha|^4}{2} + \frac{|\beta|^4}{2} + 2|\alpha\beta|^2\right]^{-(1/2)}, \tag{9}$$

and the state

$$|B\,\text{triplet}\rangle_{L1R1} \equiv \frac{1}{\sqrt{2}}(|B\uparrow\rangle_{L1}|B\downarrow\rangle_{R1} + |B\downarrow\rangle_{L1}|B\uparrow\rangle_{R1}) \tag{10}$$

is a triplet state, the antibunching result of Bob's measurement with detectors $L1$ and $R1$ on the first pair of particles.

The probability p_1 of having antibunching and thus getting state $|\varphi_1\rangle^n$ (in an ideal measurement) is

$$p_1 = \frac{|\alpha|^4}{2} + \frac{|\beta|^4}{2} + 2|\alpha\beta|^2 = N_1^{-2}, \tag{11}$$

because the first two terms have a probability $\frac{1}{2}$ of antibunching, while the other two have probability 1 (we normalize these probabilities by $|\alpha|^2$, $|\beta|^2$, and $|\alpha\beta|^2$, the probabilities of obtaining the corresponding results). Thus, the state $|\varphi_1\rangle$ differs from $|\varphi_0\rangle$ in the first two terms (we "normalize" their amplitudes α^2 and β^2 by the factor $\frac{1}{\sqrt{2}}$).

After obtaining the antibunching result, we can now let the second pair of Bob's particles ($\ell 2$ and $r2$) pass through another 50/50 beam splitter and perform the same measurement, discarding again the bunching results.

After repeating the same procedure n times, we get the state

$$|\varphi_n\rangle^n = N_n\bigg[\frac{\alpha^2}{2^{n/2}}|A\tilde{\uparrow}\rangle_L^n|A\tilde{\uparrow}\rangle_R^n|B\,\text{triplet}\rangle^{\otimes n} + \frac{\beta^2}{2^{n/2}}|A\tilde{\downarrow}\rangle_L^n|A\tilde{\downarrow}\rangle_R^n|B\,\text{triplet}\rangle^{\otimes n} + \alpha\beta(|A\tilde{\uparrow}\rangle_L^n|A\tilde{\downarrow}\rangle_R^n|B\tilde{\uparrow}\rangle_L^n|B\tilde{\uparrow}\rangle_R^n + |A\tilde{\downarrow}\rangle_L^n|A\tilde{\uparrow}\rangle_R^n|B\tilde{\downarrow}\rangle_L^n|B\tilde{\downarrow}\rangle_R^n)\bigg], \quad (12)$$

with

$$N_n = \left[\frac{|\alpha|^4}{2^n} + \frac{|\beta|^4}{2^n} + 2|\alpha\beta|^2\right]^{-(1/2)}. \quad (13)$$

The probability of getting $|\varphi_n\rangle$ from $|\varphi_{n-1}\rangle$ is

$$p_n = \left(\frac{N_{n-1}}{N_n}\right)^2. \quad (14)$$

So, the total probability for getting $|\varphi_n\rangle^n$ from $|\varphi_0\rangle^n$ is

$$\tilde{p}_n = \prod_{i=1}^n p_i = N_n^{-2} \xrightarrow{(n\to\infty)} 2|\alpha\beta|^2 \equiv \tilde{p}\,. \quad (15)$$

After applying the same procedure "infinitely many times" (to an "infinitely large state"), we obtain the state

$$|\varphi_\infty\rangle \equiv \lim_{n\to\infty}|\varphi_n\rangle^n = \lim_{n\to\infty}\frac{1}{\sqrt{2}}(|A\tilde{\uparrow}\rangle_L^n|A\tilde{\downarrow}\rangle_R^n|B\tilde{\uparrow}\rangle_L^n|B\tilde{\uparrow}\rangle_R^n + |A\tilde{\downarrow}\rangle_L^n|A\tilde{\uparrow}\rangle_R^n|B\tilde{\downarrow}\rangle_L^n|B\tilde{\downarrow}\rangle_R^n). \quad (16)$$

The total probability is then

$$p = \frac{\tilde{p}}{2} = |\alpha\beta|^2. \quad (17)$$

Now note that even if we had, instead of nonabsorbing, any kind of path detectors, our protocol would still work assuming we perform a path measurement on $n - 1$ particles. We would then have absorbed all these $n - 1$ particles, and their state would be replaced by the vacuum state $|0\rangle$. In the $n \to \infty$ limit this will factorize out to give the final maximally entangled state:

$$|\varphi_\infty\rangle = \lim_{n\to\infty}\frac{1}{\sqrt{2}}(|A\tilde{\uparrow}\rangle_L^n|A\tilde{\downarrow}\rangle_R^n|B\uparrow\rangle_L|B\uparrow\rangle_R + |A\tilde{\downarrow}\rangle_L^n|A\tilde{\uparrow}\rangle_R^n|B\downarrow\rangle_L|B\downarrow\rangle_R). \quad (18)$$

The probability p given by Eq. (17) provides us with a measure of efficiency of our entanglement concentration protocol. It is the amount of entanglement in e-bits that we can extract from a single pair of entangled particles in the initial state $|\phi\rangle^n$ given by Eq. (3), in the limiting case:

$$E^\infty(|\phi\rangle^n) = p, \qquad (n \to \infty). \quad (19)$$

Bosons.—In this case, the procedure is almost the same as for fermions, but this time we discard the antibunching results and keep the bunching ones.

After Bob measures the first pair of particles, we obtain a state similar to (8), but where the particles will now be either both in $L1$ or either both in $R1$. In other words, instead of the triplet state $|B\,\text{triplet}\rangle_{L1R1}$, we have the term $\frac{1}{\sqrt{2}}(|B\uparrow\rangle_{L1}|B\downarrow\rangle_{L1} + |B\uparrow\rangle_{R1}|B\downarrow\rangle_{R1})$, while instead of $|B\tilde{\uparrow}\rangle_L^n|B\tilde{\uparrow}\rangle_R^n$ we have $\frac{1}{\sqrt{2}}(|B\tilde{\uparrow}\rangle_{L1}|B\tilde{\uparrow}\rangle_{L1} + |B\tilde{\uparrow}\rangle_{R1}|B\tilde{\uparrow}\rangle_{R1}) \otimes |B\tilde{\uparrow}\rangle_L^{n-1}|B\tilde{\uparrow}\rangle_R^{n-1}$, and similarly for the last term in (8).

The probability of getting this state is again given by (11), $p_1 = N_1^{-2}$.

By reversing the selection of the path measurements performed by Bob (i.e., by now discarding the antibunching instead of the bunching results), we have established a symmetry between the bosonic and the fermionic protocols. The general state (12) and its probability (14) are in fact the "same" (isomorphic), as well as the total probability (17), $p = |\alpha\beta|^2$, and the efficiency (19):

$$E^\infty(|\phi\rangle^n) = p, \qquad (n \to \infty). \quad (20)$$

Note that for both protocols (for fermions and for bosons) the unitary transformation $U_\pi \equiv \bigotimes_{i=1}^n U_\pi^i$ is crucial. Since states $|\varphi\rangle^n$ and $|\varphi_0\rangle^n = U_\pi|\varphi\rangle^n$, given by Eqs. (5) and (7), respectively, are isomorphic, one might expect the results to be the same regardless of U_π being applied or not. This turns out to be wrong, as applying the protocol directly to $|\varphi\rangle^n$ will yield the following state:

$$|\varphi_\infty\rangle \equiv \lim_{n\to\infty}|\varphi_n\rangle = \lim_{n\to\infty}\frac{1}{\sqrt{|\alpha|^4 + |\beta|^4}}(\alpha^2|A\tilde{\uparrow}\rangle_L^n|A\tilde{\uparrow}\rangle_R^n|B\tilde{\downarrow}\rangle_L^n|B\tilde{\downarrow}\rangle_R^n + \beta^2|A\tilde{\downarrow}\rangle_L^n|A\tilde{\downarrow}\rangle_R^n|B\tilde{\uparrow}\rangle_L^n|B\tilde{\uparrow}\rangle_R^n), \quad (21)$$

in general less entangled than the initial one.

In the protocols we have presented here, local operations are performed on one side only (Bob). Classical communication comes about only once, when we have one way communication from Bob to Alice at the end of the whole scheme. Of course, we can slightly change our protocols by allowing Alice to apply the same procedure on her side. This would require two-way communication and would have half the time complexity of the protocol, but the amount of entanglement distilled would be the same. Also, note that both Alice and Bob could perform the operations on the $n - 1$ particles, each one on their side, using either one beam splitter sequentially or $n - 1$ beam splitters in parallel. The latter case has a lower time complexity than the former, but requires more resources (higher space complexity).

The efficiency $|\alpha\beta|^2$ of our protocol happens to be lower than both the efficiency $2|\alpha|^2$ of the procrustean method and the efficiency $-|\alpha|^2 \operatorname{Log}|\alpha|^2 - |\beta|^2 \operatorname{Log}|\beta|^2$ of the standard asymptotic entanglement concentration procedure [3]. However, our target is *not* to propose a more efficient *alternative* entanglement concentration, but to explicitly demonstrate that it is *possible* to do entanglement concentration by using *only* the effects of particle statistics, without resort to an explicit controlled operation between spins or between spin and path. In all physical implementations of entanglement concentration or purification protocols to date [10,11], a polarization beam splitter or a polarization dependent filter (which accomplish a controlled operation on path conditional on spin) have been used.

It is important to note here that our mechanism will work even if the basis states $|\uparrow\rangle$ and $|\downarrow\rangle$, were rotated in a plane. This is because the only unitary rotation we use is U_π, which will flip all spins in one plane. The rest of the protocol uses particle statistics, which is basis independent. On the other hand, for standard entanglement concentration protocols [2,3], it is necessary to know the basis.

In this paper we have presented an entanglement concentration scheme which uses *only* the effects of quantum statistics. Although the efficiency of the protocol is the same for both fermions and bosons, the protocol itself is slightly different depending on the nature of the particles. This brings forth a fundamental difference between these two types of particles in terms of their information processing power. Recent experiments such as [13,14] suggest that it would be possible to test our results in the near future. It would also be interesting to investigate the possibility of entanglement distillation of mixed states using *solely* statistical effects. Future work will comprise investigating more extensive quantum information processing procedures.

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