

# Indistinguishable particles in quantum mechanics: an introduction

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In this article, we discuss the identity and indistinguishability of quantum systems and the consequent need to introduce an extra postulate in Quantum Mechanics to correctly describe situations involving indistinguishable particles. This is, for electrons, the Pauli Exclusion Principle, or in general, the Symmetrization Postulate. Then, we introduce fermions and bosons and the distributions respectively describing their statistical behaviour in indistinguishable situations. Following that, we discuss the spin-statistics connection, as well as alternative statistics and experimental evidence for all these results, including the use of bunching and antibunching of particles emerging from a beam splitter as a signature for some bosonic or fermionic states.

## 1. An extra postulate is required

I believe most physicists would consider that the postulates (or at least the properties they embody) concerning the superposition, evolution and measurement of quantum states cover the essence of Quantum Mechanics, the theory that is at the basis of current fundamental Physics and gives us such an accurate description of Nature at the atomic scale. Yet, if the theory was only based on these postulates (or properties), its descriptive power would be almost zero and its interest, if any, would be mainly mathematical. As soon as one wants to describe matter, one has to include an extra postulate: *Pauli's Exclusion Principle*. One of its usual formulations, equivalent to the one proposed originally by Wolfgang Pauli in 1925 [1], is the following:

**Pauli's Exclusion Principle**—*No two electrons can share the same quantum numbers.*

This principle refers to electrons, which constitute a significant (but not the whole) part of matter, and is crucial in helping us explain a wide range of phenomena, including:

- (a) the electronic structure of atoms and, as a consequence, the whole Periodic Table;
- (b) the electronic structure of solids and their electrical and thermal properties;
- (c) the formation of white dwarfs, where the gravitational collapse of the star is halted by the pressure resulting from its electrons being unable to occupy the same states;
- (d) the repulsive force that is part of the *ionic bond* of molecules and puts a limit to how close the ions can get (e.g. 0.28 nm between  $\text{Na}^+$  and  $\text{Cl}^-$  for solid sodium chloride), given the restrictions to the states the overlapping electrons can share.

We thus see how Pauli's insight when proposing the Exclusion Principle was fundamental for the success of Quantum Mechanics. Although he made many other important contributions to Physics, it was for this one that he was awarded the Nobel prize in 1945.

Historically, it is also interesting to note that this happened before Samuel Goudsmit and Georg Uhlenbeck introduced the idea of *electron spin* [2,3] later in 1925. In 1913, Niels Bohr presented his model for the electronic structure of the atom to explain the observed discrete energy spectra of hydrogen and other elements [4]: the electrons fly around the positive nucleus in circular orbits<sup>†</sup>

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<sup>†</sup>A wrong idea, as we shall discuss later.

with quantized angular momentum. This quantization restricts the possible orbits to a discrete set, each corresponding to an energy level of the atom. This model was then improved during the following decade, mainly by Arnold Sommerfeld and Alfred Landé, rendering it more sophisticated, trying to make it able to account for the multiplet structure of spectral lines, including for atoms in electric and magnetic fields. In 1922, Pauli joins the effort (actually, it was rather a competition) to find an explanation for the then-called *anomalous Zeeman effect*, a splitting of spectral lines of an atom in a magnetic field that was different from the already known Zeeman splitting (see [5] for a technical historical account on this competition). Another puzzle at the time, identified by Bohr himself, was the following: how to explain that in an atom in the ground state the electrons do not all populate the orbit closest to the nucleus (corresponding to the lowest energy level) [6]? These two problems led Pauli to postulate, towards the end of 1924, a new property for the electron—‘*a two-valuedness not describable classically*’ [7]—and soon after the Exclusion Principle [1] as fundamental rules for the classification of spectral lines. But Pauli did not present any model for this extra degree of freedom of the electrons. A few months later, Goudsmit and Uhlenbeck introduced the idea of an intrinsic angular momentum of  $\frac{1}{2}\hbar$  for the electron, finding not only a definite explanation for the *anomalous Zeeman effect*, but also establishing since then a connection between spin and the Exclusion Principle, a connection whose depth they could not guess.

Pauli’s Exclusion Principle remains as a postulate, for Pauli’s own dissatisfaction, as he expressed in his Nobel prize acceptance lecture in 1946:

*‘Already in my original paper I stressed the circumstance that I was unable to give a logical reason for the exclusion principle or to deduce it from more general assumptions. I had always the feeling, and I still have it today, that this is a deficiency.’* [8]

In any case, as inexplicable as it may be, Pauli’s Exclusion Principle seems to beg for a generalization. In fact, it was soon realized that other particles apart from electrons suffer from the same inability to share a common quantum state (e.g. protons). More surprising was the indication that some particles seem to obey the exact opposite effect, being—under certain circumstances—forced to share a common state, as for instance photons in the stimulated emission phenomenon, thus calling for a much more drastic generalization of Pauli’s Principle, as we shall see.

## 2. Identity and indistinguishability

We saw that Pauli’s Exclusion Principle intervenes in a wide range of phenomena, from the chemical bond in the salt on

our table to the formation of stars in distant galaxies. This is because it applies to electrons and we consider all electrons in the universe to be *identical*, as well as any other kind of quantum particles:

**Identical particles**—*Two particles are said to be identical if all their intrinsic properties (e.g. mass, electrical charge, spin, colour, . . .) are exactly the same.*

Thus, not only all electrons are identical, but also all positrons, photons, protons, neutrons, up quarks, muon neutrinos, hydrogen atoms, etc. They each have the same defining properties and behave the same way under the interactions associated with those properties. This brings us to yet another purely quantum effect, that of *indistinguishable* particles.

Imagine we have two completely identical classical objects, that we cannot differentiate in any way. Should we give them arbitrary labels, we could always—at least in principle—keep track of which object is which by following their respective *trajectories*. But we know that in quantum mechanics we must abandon this classical concept. The best information we can get about some particle’s location without measuring it (and thus disturbing it) is that it has a certain probability of being in a particular position in space, at a given moment in time. This information is contained in the particle’s *spatial state*, for instance given by

$$|\psi\rangle_V = \int \psi(\mathbf{r}, t) |\mathbf{r}\rangle d^3 r, \quad (1)$$

where the vectors  $|\mathbf{r}\rangle$ , each representing the state corresponding to a particular position of the particle in the three-dimensional Euclidean space, constitute an orthonormal (continuous) basis of the Hilbert space associated with the position degree of freedom. The coefficient  $\psi(\mathbf{r}, t)$ , also known as the particle’s *wave function*, contains the probabilistic information about the location of the particle, the only information available to us prior to a measurement. The probability of finding the particle in position  $\mathbf{r}_0$  at a time  $t$  is given by

$$P(\mathbf{r}_0, t) = |\langle \mathbf{r}_0 | \psi \rangle_V|^2 = |\psi(\mathbf{r}_0, t)|^2. \quad (2)$$

Note that there is usually a volume  $V$  outside which  $\psi(\mathbf{r}, t)$  quickly falls off to zero asymptotically. We associate the *spread* of the wave function to this volume  $V$ , which can evolve in time. Finally, recall that because of Heisenberg’s uncertainty relations we cannot simultaneously measure the particle’s position and its momentum with an arbitrary precision (see, for instance, [9]).

How can we then distinguish identical particles? Their possibly different internal states are not a good criterion,

as the dynamics can in general affect the internal degrees of freedom of the particles. The same is valid for their momentum or other dynamical variables. But their spatial location can actually be used to distinguish them, as shown in figure 1. Let us imagine we have two identical particles, one in Alice's possession and the other with Bob. If these two parties are kept distant enough so that the wave functions of the particles practically never overlap (during the time we consider this system), then it is possible to keep track of the particles just by keeping track of the classical parties. This situation is not uncommon in quantum mechanics. If, on the other hand, the wave functions do overlap at some point, then we no longer know which particle is with which party, as shown in figure 2. And if we just do not or cannot involve these classical parties at all, then it is in general also impossible to keep track of identical particles. In both these cases, the particles become completely indistinguishable, they are identified by completely arbitrary labels, with no physical meaning (as opposed to *Alice* and *Bob*). In these situations very interesting new phenomena arise.

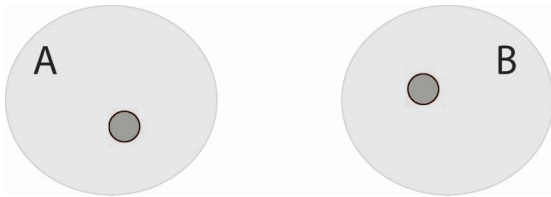


Figure 1. This figure represents two distant identical particles, as well as the spread of their respective wave functions. Each one of them occupies a distinct region of space, arbitrarily labelled *A* and *B*, thus allowing us to distinguish these identical particles, just as in a classical case.

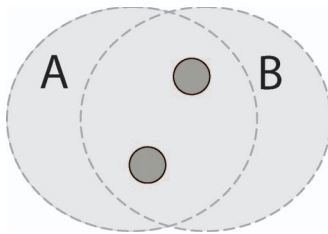


Figure 2. This figure represents two identical particles in a situation where their respective wave functions overlap. It is no longer unambiguous which region of space each particle occupies. The particles are indistinguishable—a purely quantum effect—and must now obey the Symmetrization Postulate: there are symmetry restrictions to the states describing the joint system.

### 3. Symmetries, fermions and bosons

Let us consider the following example. Imagine that we have two electrons in an indistinguishable situation, e.g. as the one described in figure 2. We can arbitrarily label them 1 and 2. We also know that one of the particles has spin up along the  $z$  direction and the other has spin down along the same direction. How shall we describe the (spin) state of our system? One possibility would be to consider the vector:

$$|\mu\rangle = |\uparrow\rangle_1 |\downarrow\rangle_2, \quad (3)$$

where  $|\uparrow\rangle_i$  and  $|\downarrow\rangle_i$  represent the two opposite spin components along  $z$  for each particle  $i$ , and  $\{|\uparrow\rangle_i, |\downarrow\rangle_i\}$  constitutes an orthonormal basis of the 2-dimensional Hilbert space. But, since the particles are indistinguishable, we could permute their labels and the state of the system could equally be described by the vector:

$$|\nu\rangle = |\downarrow\rangle_1 |\uparrow\rangle_2. \quad (4)$$

Note that the state of the system is the same, but we have two different vectors that can validly describe it. In fact, taking into account the superposition principle, a linear combination of  $|\mu\rangle$  and  $|\nu\rangle$  will also be a possible description of our state:

$$|\kappa\rangle = \alpha |\uparrow\rangle_1 |\downarrow\rangle_2 + \beta |\downarrow\rangle_1 |\uparrow\rangle_2, \quad (5)$$

where  $\alpha, \beta \in \mathbb{C}$  are chosen such that  $|\alpha|^2 + |\beta|^2 = 1$ . So, we actually have an infinity of different mathematical descriptions for the same physical state. This is a consequence of the indistinguishability of particles and is known as *exchange degeneracy*. How can we then decide which of the above vectors is the correct description of our state, i.e. which one will allow us to make correct predictions about measurements or the evolution of the system? Should it be  $|\kappa\rangle$ , and for which particular values of  $\alpha$  and  $\beta$ ? Also, note that our example could be generalized to more and other species of particles: the exchange degeneracy appears whenever we deal with indistinguishable particles. The problem of finding the correct and unambiguous description for such systems is thus very general and requires the introduction of a new postulate for quantum mechanics: the Symmetrization Postulate.

**Symmetrization Postulate**—*In a system containing indistinguishable particles, the only possible states of the system are the ones described by vectors that are, with respect to permutations of the labels of those particles:*

- (i) *either completely symmetrical—in which case the particles are called bosons;*
- (ii) *either completely antisymmetrical—in which case the particles are called fermions.*

It is this information that will allow us to lift the exchange degeneracy. But first, let us consider a general example to discuss the concepts and issues introduced by this postulate.

Imagine we have a system  $\Sigma$  of  $N$  indistinguishable particles, to which we associate the Hilbert space  $\mathcal{H}_N \equiv \mathcal{H}^{\otimes N}$ . We arbitrarily label the particles with numbers from 1 to  $N$ . The state of the system can be described by

$$|\Omega\rangle_N \equiv |\alpha\rangle_1 \otimes |\beta\rangle_2 \otimes \cdots \otimes |\xi\rangle_i \otimes \cdots \otimes |\tau\rangle_N, \quad (6)$$

where  $|\xi\rangle_i$  arbitrarily indicates that particle  $i$  is in state  $|\xi\rangle \in \mathcal{H}$ . But the above postulate imposes some symmetries. Let us first define the following terms.

A *completely symmetric* vector  $|\psi_S\rangle$  is a vector that remains invariant under permutations of its  $N$  labels (or components), such that

$$\hat{P}_N |\psi_S\rangle = |\psi_S\rangle \quad (7)$$

for any permutation  $P_N$  of those  $N$  different labels. In  $|\Omega\rangle_N$ , and in most of our other examples, these will just be the integers from 1 to  $N$ . Also, note that there are  $N!$  such permutations.

Similarly, a *completely antisymmetric* vector  $|\psi_A\rangle$  is a vector that satisfies:

$$\hat{P}_N |\psi_A\rangle = \varepsilon_{P_N} |\psi_A\rangle \quad (8)$$

for any permutation  $P_N$ , and where:

$$\varepsilon_{P_N} = \begin{cases} +1 & \text{if } P_N \text{ is an even permutation,} \\ -1 & \text{if } P_N \text{ is an odd permutation.} \end{cases} \quad (9)$$

To impose these symmetries to a vector, we can define the *symmetrizer* and *antisymmetrizer* operators, respectively given by

$$\hat{S} \equiv \frac{1}{N!} \sum_{P_N} \hat{P}_N \quad (10)$$

and

$$\hat{A} \equiv \frac{1}{N!} \sum_{P_N} \varepsilon_{P_N} \hat{P}_N, \quad (11)$$

and where the sums are taken over the  $N!$  possible permutations  $P_N$  of a set of  $N$  elements. We can then apply these operators to  $|\Omega\rangle_N$  to obtain a completely symmetric vector:

$$|\Omega_S\rangle_N \equiv \hat{S}|\Omega\rangle_N, \quad (12)$$

as well as a completely antisymmetric vector:

$$|\Omega_A\rangle_N \equiv \hat{A}|\Omega\rangle_N, \quad (13)$$

one of which will be the correct description of our system  $\Sigma$ , in accordance to the Symmetrization Postulate. Note that  $\hat{S}$

is the projection operator onto the completely symmetric subspace of  $\mathcal{H}_N$ , i.e. the Hilbert space spanned by all the independent completely symmetric vectors of  $\mathcal{H}_N$ , which we shall call  $\mathcal{H}_S$ . Analogously,  $\hat{A}$  is the projector onto the completely antisymmetric subspace  $\mathcal{H}_A$ . In general, we have the relation:

$$\mathcal{H}_N = \mathcal{H}_S \oplus \mathcal{H}_A \oplus \mathcal{H}_m, \quad (14)$$

where  $\mathcal{H}_m$  is a subspace of  $\mathcal{H}_N$  with mixed symmetry. We thus see that, for systems of indistinguishable particles, the Symmetrization Postulate restricts the Hilbert space of the system. The only acceptable vectors to describe the system must lie in either the completely symmetric or in the completely antisymmetric subspaces. But how do we know which one of these two exclusive possibilities to choose? In particular, coming back to the example system  $\Sigma$ , which of the vectors  $|\Omega_S\rangle_N$  and  $|\Omega_A\rangle_N$  represents the state of our system? This is something that depends on the nature of the particles, if they are either bosons or fermions respectively. And to which of these two classes a particle belongs is something that ultimately can only be determined experimentally.

### 3.1 Fermions

We call *fermions* identical particles that, in an indistinguishability situation, can only be found in antisymmetric states, i.e. described by vectors in the system's antisymmetric subspace. Such states can be constructed using the antisymmetrization operator  $\hat{A}$  defined in equation (11). It is interesting to note that the sum of permutations with alternating sign can formally be obtained using a determinant. This is known as the *Slater determinant* and offers us a practical way to construct completely antisymmetric vectors. For instance,  $|\Omega_A\rangle_N$  of our example can be calculated the following way:

$$\begin{aligned} |\Omega_A\rangle_N &= \hat{A}|\Omega\rangle_N \\ &= \frac{1}{N!} \begin{vmatrix} |\alpha\rangle_1 & |\beta\rangle_1 & \cdots & |\xi\rangle_1 & \cdots & |\tau\rangle_1 \\ |\alpha\rangle_2 & |\beta\rangle_2 & \cdots & |\xi\rangle_2 & \cdots & |\tau\rangle_2 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ |\alpha\rangle_i & |\beta\rangle_i & \cdots & |\xi\rangle_i & \cdots & |\tau\rangle_i \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ |\alpha\rangle_N & |\beta\rangle_N & \cdots & |\xi\rangle_N & \cdots & |\tau\rangle_N \end{vmatrix}. \end{aligned} \quad (15)$$

Note that if there are two particles of the system in the same state, say  $|\alpha\rangle = |\beta\rangle$ , then two columns of the Slater determinant will be equal and the determinant will be zero. This means that there are no vectors to describe systems of fermions where two particles are in the same state. This is just as the Exclusion Principle which we

introduced for electrons in section 1, but now for a much broader set of particles. We conclude that the Symmetrization Postulate not only includes, but actually generalizes Pauli's Exclusion Principle to other particles than electrons. Moreover, we can now say that electrons are fermions. This information allows us to apply the postulate to our example of the two electrons with anti-aligned spins discussed in the beginning of section 3 and thus finally lift the respective exchange degeneracy. Let us assume, for the sake of the argument, that in this example the total spatial wave function of the particles is symmetrical. The correct description of this system is then given by the vector in equation (5) with the following choice of coefficients to make it, together with the full state, completely antisymmetric:

$$|\kappa_A\rangle \equiv \frac{1}{2^{1/2}} (|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2). \quad (16)$$

We see that, even for electrons, the Symmetrization Postulate gives us more information than the Exclusion Principle.

We now have a more physical and operational definition of fermions: particles that, when indistinguishable, can never be in the same state. This restriction has very clear consequences when we study the statistical properties of quantum systems of many identical particles. All fermions follow the same distribution for the average number of particles in a certain quantum state, say  $\ell$ , in function in terms of the parameters of the system:

$$\langle n_\ell \rangle = \frac{1}{\exp[(E_\ell - \mu)/k_B T] + 1}, \quad (17)$$

where  $\mu$  is the chemical potential per particle,  $E_\ell$  is the energy of the particle in state  $\ell$ ,  $T$  is the temperature of the system and  $k_B \simeq 1.38 \times 10^{-23} \text{ J K}^{-1}$  is Boltzmann's constant. This is known as *Fermi-Dirac's statistical distribution* and is a direct consequence of the antisymmetry of the fermionic states (see, for example, [10] for the proof). Note that we always have  $\langle n_\ell \rangle \leq 1$ , as we would expect. This distribution was first calculated for electrons by Enrico Fermi [11] in 1926 and its more general relations with quantum mechanics were established soon after by Paul Dirac [12]. This distribution plays a central role in quantum statistics. It is, for example, fundamental for describing the electronic structure of solids and their electrical and thermal properties. Note also that in the limit where particles become distinguishable (e.g. because of a lower density, a larger separation between energy levels, etc.) as thermal fluctuations become more important than the quantum ones, we recover the classical Boltzmann distribution.

We thus have two ways to decide whether a given type of particles are fermions: they explicitly obey the Exclusion Principle (more easily observable in systems with few particles) and they follow the Fermi-Dirac distribution (a criterion more adequate for systems of many particles).

In any case, an experimental proof is necessary. It has also been observed that all fermions have half-integer spin. It should be noted that this surprising and useful property is not part of the definition of fermions, or at least does not need to be. The connection between spin and statistics will be further discussed in section 3.3. Particles that are nowadays known to be fermions include: electrons and in fact all leptons, quarks, protons, neutrons, baryons in general,  $^3\text{He}$ , etc.

Finally, note that the symmetry requirements of the Symmetrization Postulate are, in the second quantization formalism<sup>†</sup>, replaced by imposing certain algebraic relations on the fermionic creation and annihilation operators, respectively denoted  $\hat{a}_i^\dagger$  and  $\hat{a}_i$ . These are defined by

$$\begin{cases} \hat{a}_i^\dagger |n_1, \dots, n_{i-1}, 0_i, n_{i+1}, \dots\rangle = (-1)^{p_i} |n_1, \dots, n_{i-1}, 1_i, n_{i+1}, \dots\rangle \\ \hat{a}_i^\dagger |n_1, \dots, n_{i-1}, 1_i, n_{i+1}, \dots\rangle = 0 \end{cases} \quad (18)$$

and

$$\begin{cases} \hat{a}_i |n_1, \dots, n_{i-1}, 1_i, n_{i+1}, \dots\rangle = (-1)^{p_i} |n_1, \dots, n_{i-1}, 0_i, n_{i+1}, \dots\rangle \\ \hat{a}_i |n_1, \dots, n_{i-1}, 0_i, n_{i+1}, \dots\rangle = 0 \end{cases} \quad (19)$$

with  $p_i = \sum_{k < i} N_k$ , where  $N_k$  is the eigenvalue of the number operator  $\hat{N}_k \equiv \hat{a}_k^\dagger \hat{a}_k$ . The condition of complete antisymmetry for vectors describing indistinguishable fermions is then replaced by the following anti-commutation relation on the creation and annihilation operators describing the system:

$$[\hat{a}_i, \hat{a}_j^\dagger]_+ \equiv \hat{a}_i \hat{a}_j^\dagger + \hat{a}_j^\dagger \hat{a}_i = \delta_{ij} \hat{I}, \quad (20)$$

where  $i$  and  $j$  are sets of labels for (in general) different modes,  $\delta_{ii} = 1$  and  $\delta_{ij} = 0$  for  $i \neq j$ . The two formalisms are equivalent.

### 3.2 Bosons

We call *bosons* identical particles that, in an indistinguishability situation, can only be found in symmetric states, i.e. described by vectors in the system's symmetric subspace. Such states can be constructed using the symmetrization operator  $\hat{S}$  defined in equation (10). The average number of bosons in a certain quantum state, say  $\ell$ , is given by the *Bose-Einstein statistical distribution* [10]:

$$\langle n_\ell \rangle = \frac{1}{\exp[(E_\ell - \mu)/k_B T] - 1}, \quad (21)$$

<sup>†</sup>The reader unfamiliar with this formalism is referred to appendix A for a brief introduction.

where  $\mu$  is the chemical potential per particle and  $\mu < 0$ ,  $E_\ell$  is the energy of the particle in state  $\ell$ ,  $T$  is the temperature of the system and  $k_B$  is Boltzmann's constant. Formally, this distribution differs only by a sign when compared with Fermi–Dirac's one, given by equation (17), but the differences are actually great: in this case,  $(n_\ell)$  is unbounded. For bosons there is no Exclusion Principle. Firstly, an arbitrary number of them can occupy the same state. Secondly, in certain cases the Symmetrization Postulate actually forces the particles to share a common state, other alternatives being incompatible with the symmetry requirements. It is almost like bosons follow some kind of *aggregation principle*, opposing Pauli's one.

The Bose–Einstein distribution was first proposed by Satyendranath Bose in 1924 for a gas of photons [13] and very soon after generalized by Albert Einstein to an ideal monoatomic gas [14,15]. Einstein noticed then that, under a certain critical temperature, a fraction of the particles would gather in the energy ground state [15]; this fraction grows as the temperature decreases and eventually includes all the particles when the temperature is zero. This phase transition is known as the *Bose–Einstein condensation* and has only been directly observed (with non-interacting particles) in 1995—seventy years after Einstein predicted it—using rubidium atoms at 20 nK [16]. As in the case of fermions, the Bose–Einstein distribution tends to Boltzmann's as particles approach the classical regime.

For bosons there is also a connection between spin and statistics. In fact, we believe that all particles of integer spin (including 0) are bosons: photons, gluons,  $W^\pm$ ,  $Z^0$ , mesons,  $^4\text{He}$  and any other atoms or nuclei with integer spin, etc. This property will be discussed in more detail in section 3.3.

Finally, a word about the treatment of bosons in second quantization. In this formalism we define the bosonic field creation operator  $\hat{a}_i^\dagger$  by

$$\hat{a}_i^\dagger |n_1, \dots, n_{i-1}, n_i, n_{i+1}, \dots\rangle = (n_i + 1)^{1/2} |n_1, \dots, n_{i-1}, n_i + 1, n_{i+1}, \dots\rangle, \quad (22)$$

and the bosonic field annihilation operator  $\hat{a}_i$  by

$$\hat{a}_i |n_1, \dots, n_{i-1}, n_i, n_{i+1}, \dots\rangle = n_i^{1/2} |n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots\rangle. \quad (23)$$

Note that we use exactly the same notation for the bosonic and fermionic field operators (defined in section 3.1): although this can at first seem a source of confusion, this notation will be useful later. But, of course, one should keep in mind at all times that these are different operators by definition, with different commutation properties. In particular, the requirement of complete symmetry for bosons is obtained imposing the following commutation condition on the creation and annihilation operators describing the system:

$$[\hat{a}_i, \hat{a}_j^\dagger]_- \equiv \hat{a}_i \hat{a}_j^\dagger - \hat{a}_j^\dagger \hat{a}_i = \delta_{ij} \hat{I}, \quad (24)$$

where  $i$  and  $j$  are sets of labels for (in general) different modes,  $\delta_{ii} = 1$  and  $\delta_{ij} = 0$  for  $i \neq j$ .

### 3.3 The spin-statistics connection

We saw that to determine whether a given particle is a fermion or a boson, we need to investigate its statistical behaviour in the presence of (at least one) other identical particle, when they are all indistinguishable. Of course, we can expect that if a composite particle is made of bosons, or of an even number of fermions, then it should be a boson. And if, on the other hand, the particle is composed of an odd number of fermions, then it should be a fermion itself. But for the particles we believe to be fundamental, a direct study of their statistical nature may be required. From the experimental point of view, such a study can represent quite a difficult challenge. For instance, we may be dealing with rare particles, difficult to observe (e.g. neutrinos), to find in free form (e.g. quarks and gluons) or even to produce (e.g. the *top* quark), or particles that are just too unstable or short-lived (e.g. the  $\tau$  lepton) or yet to be observed (e.g. the graviton). Indirect methods could also help us reach a conclusion, but before any of that a simple and intriguing property can actually come to our rescue: the *spin-statistics connection*.

**Spin-Statistics Theorem**—*Particles with integer spin are bosons. Particles with half-integer spin are fermions.*

This is not only a widely known empirical rule in Physics, but in fact a theorem, even if its proofs are not all completely clear and free from controversy. Thanks to it, it is very easy to determine whether some particle is either a fermion or a boson. In particular, this criterion works also for composite particles and is consistent with the previous conclusion that particles composed of an arbitrary number of bosons and/or of an even number of fermions are bosons and that particles composed of an odd number of fermions are fermions. It is quite surprising to find such a connection between the spin of a particle and its statistical nature, a connection whose origins I think are still not well understood.

The first and reference proof of the Spin-Statistics Theorem is usually and fairly attributed to a 1940 article by Pauli [17], despite some earlier contributions towards this problem (see [18] for a historical and technical account of those works). Pauli's proof, referring only to free particles, is based on two (reasonable) assumptions:

- (a) by assuming that the energy must be a positive quantity, he concludes that particles with half-integer spins cannot be bosons;

- (b) by assuming *microcausality*<sup>†</sup>, he concludes that particles with integer spin cannot be fermions.

Markus Fierz had just proven a year earlier that, under these same assumptions, particles with integer spin could be bosons and particles with half-integer spin could be fermions [19], and with those results Pauli could conclude his proof. Pauli wanted to present the spin-statistics connection as a direct consequence of Special Relativity, but his *negative* and *asymmetric* proof depended also on the condition of positive energy. Several other proofs appeared over the following decades, more or less along the lines of Pauli's work and always in the context of Relativistic Quantum Mechanics, and in particular of Quantum Field Theory (see [18] and references therein). Note also Steven Weinberg's proof [20] based solely on the Lorentz invariance of the *S-matrix*<sup>‡</sup>, breaking with some traditions in the approach to the theorem [22]. These (and other similar) works were a source of debates and brought some extensions and clarifications to details of other proofs, but there was no real advance in our understanding of this connection. To put it in Richard Feynman's words:

*'It appears to be one of the few places in physics where there is a rule which can be stated very simply, but for which no one has found a simple and easy explanation. [...] This probably means that we do not have a complete understanding of the fundamental principle involved.'* [23]

If this relation between spin and statistics is indeed a relativistic effect, or at least if the proof of the respective theorem relies on Special Relativity, then the spin-statistics connection has to be introduced in (non-relativistic) Quantum Mechanics as a postulate. Recently, some non-relativistic proofs have been proposed [24–26], but the claims are not uncontroversial (see [26]). It remains that the connection between spin and statistics is an empirical rule, and for now that is probably how it should be introduced in the context of Quantum Mechanics.

Research on the connection between spin and statistics continues also in other directions, in particular in the study of particles obeying alternatives statistics. These will be briefly discussed in section 4. Note also that in 1974 a new fundamental symmetry was proposed [27], *supersymmetry*, which transforms bosons into fermions and vice versa, providing a framework for the unification of gravity with

the other interactions (for more details, see for example [28]). To this date, no experimental evidence of supersymmetric particles has been found though [29]. Nor, in fact, any evidence of violations of the predictions of the Spin-Statistics Theorem—maybe our best and clearest proof so far that it holds.

### 3.4 Discussion of the Symmetrization Postulate

Historically, Werner Heisenberg was the first to show—in 1926, in the context of wave mechanics [30,31]—that the states of a system of identical particles are restricted to specific symmetry classes which cannot be transformed into each other. To decide which particles should be associated to a particular symmetry is something that must ultimately be determined by observation. The Symmetrization Postulate matches the study of such symmetries with our empirical knowledge: as far as we know today, there are two classes of particles in Nature according to their collective behaviour in indistinguishable situations. These are, of course, bosons and fermions: no particles have been found so far that under the same circumstances could be described by vectors that are neither symmetrical nor antisymmetrical (see section 4). It is important to note that none of this could have been deduced from the other postulates of Quantum Mechanics. Yet, the Symmetrization Postulate is rarely evoked. In textbooks, it is never presented with the postulates of Quantum Mechanics, but rather postponed for a final chapter about identical particles, if it appears at all as such. Pedagogical and epistemological issues aside, such an approach gives in my opinion an incomplete picture of the theory, both formally and physically, responsible for some ignorance and misunderstandings. It is true that the Symmetrization Postulate applies only to indistinguishable particles and is, in that sense, less general than the other postulates of Quantum Mechanics. Moreover, it imposes a limitation on the *Hilbert (or state) space postulate*, by restricting the state space of the system to its completely symmetric or antisymmetric subspaces. But, on the other hand, it is also true that what makes Quantum Mechanics so interesting is that it can describe matter (and fields) around us, and for that, as we saw, it is necessary to introduce the Pauli Exclusion Principle, or—more generally—the Symmetrization Postulate. This is, after all, the way we formally introduce fermions and bosons in Quantum Mechanics.

The awkward status of the Symmetrization Postulate probably reflects our discomfort in trying to understand some of these issues, as Pauli mentioned already in 1946:

*'Of course in the beginning I hoped that the new quantum mechanics, with the help of which it was possible to deduce so many half-empirical formal rules in use at that time, will also rigorously deduce the*

<sup>†</sup>The usual special relativity restriction that the measurement of a physical system cannot influence another if the two are space-like separated.

<sup>‡</sup>The *S-matrix formalism* is an alternative approach to relativistic quantum physics based on the unitary S-matrix that encodes all the information on all possible scattering processes. Formally, the S-matrix is the realization of the isomorphism between the *in* and *out* Fock spaces. For more details, see for instance [21].

*exclusion principle. Instead of it there was for electrons still an exclusion: not of particular states any longer, but of whole classes of states, namely the exclusion of all classes different from the antisymmetrical one. The impression that the shadow of some incompleteness fell here on the bright light of success of the new quantum mechanics seems to me unavoidable.*' [8]

We have a very simple and elegant rule, crucial for the success of Quantum Mechanics as a theory describing Nature, undisputed and apparently free from interpretations, unlike other quantum rules. We have no clear understanding of its origin, not even within the framework of a more sophisticated theory than Quantum Mechanics<sup>†</sup>, and are thus happy (or forced) to accept it as an empirical rule, but one separated from the other postulates of the theory, even if it is not less mysterious than these, nor less fundamental as far as we know today.

Let us now analyse the statement of the Symmetrization Postulate. The principle applies only to indistinguishable particles. This is exactly how we have presented it here, although some prefer to state it as applying to identical particles in general. *A priori*, the latter approach implies that, for instance, every electron in the universe would have to be antisymmetrized with respect to all the others. And every other kind of particle would have to obey a similar requirement, adequate to its statistical nature. This picture is of course very unsatisfactory, but—conveniently—it can be shown that the (anti)symmetrization terms have a vanishing probability when we can distinguish the identical particles (see, for example, [9]). Thus, the Symmetrization Postulate ends up being applied only to indistinguishable particles. Note also that, given a system, we often consider separate vectors to describe its spatial and its internal degrees of freedom. Then, the symmetries of each vector are not independent, as they must consistently contribute to the symmetry requirement of the full vector describing the state of the system. This is a natural consequence of the postulate and is not implied by the concept of *complete symmetry*, which specifically refers to all possible permutations that can be considered. Finally, note that if a composite particle is made of bosons or of an even number of fermions then it is a boson, and if it is composed of an odd number of fermions then it is itself a fermion.

All evidence to this date points to the fact that quantum indistinguishable particles either follow Fermi–Dirac's or Bose–Einstein's statistics, something that is generally referred to as *particle* or *quantum statistics*. Accordingly, the Symmetrization Postulate, with its complete symmetry

or antisymmetry requirement, rules out the consideration of other statistics from the onset. Yet, without this restriction, Quantum Mechanics would actually allow unusual symmetries, more complex than the previous ones. As we shall see in section 4, consistent theories have been developed allowing for small deviations from the conventional statistics which might have been undetected in the experiments performed so far.

Finally, I would like to point out the interesting and possibly deep fact that, according to the Standard Model of Particle Physics, the fundamental constituents of matter—quarks and leptons—are fermions, whereas the force carriers are bosons. Note also that, within the framework of Quantum Field Theory, the equivalent to the Symmetrization Postulate is introduced by imposing local (equal-time) commutation and anti-commutation rules on the field operators, similar to the ones given by equations (24) and (20).

#### 4. Experimental evidence and alternative statistics

Particle statistics, or the existence of fermions and bosons and their respective properties, is well established, accounting for a series of different physical phenomena and having been subjected to extensive experimental corroboration. The origin of the Symmetrization Postulate itself lies in the attempt to explain experimental data, in particular the energy spectra of atoms. In 1925, Pauli proposed the Exclusion Principle (as we discussed in section 1). Soon after, Heisenberg was able to introduce it, together with spin, in the context of wave mechanics and used the symmetry properties of the electrons' wave functions (decomposed in spatial and spin wave functions) to explain the two classes of spectral lines observed for helium: *para-helium* and *ortho-helium* [31]. Here came into play the *exchange interaction*, relating the energy levels occupied by the electrons to the symmetry of their wave functions and the corresponding spatial and spin states, a purely quantum effect associated with the Pauli Principle.

As we saw, the Symmetrization Postulate generalized the Pauli Exclusion Principle to more particles than just electrons, i.e. fermions, and introduced a fundamentally new type of particle, i.e. bosons. Moreover, it applies to composite systems in general, where one single particle may be enough to change the system's statistical nature and corresponding behaviour. A paradigmatic example is given by the two isotopes of helium: <sup>3</sup>He (a fermion) and <sup>4</sup>He (a boson). The latter exhibits remarkable properties when cooled down to temperatures below 2 K: it becomes a *superfluid* liquid, with practically no viscosity and capable of flowing with no friction through tiny capillaries [32]<sup>‡</sup>.

<sup>†</sup>As opposed to the spin-statistics connection, a property that we believe can be derived in the context of relativistic Quantum Mechanics, as we saw in section 3.3.

<sup>‡</sup>Note that at even lower temperatures, below 3 mK, <sup>3</sup>He can also exhibit a superfluid phase [33,34], as the fermionic helium atoms pair up to form bosonic quasiparticles.



Particle statistics helps us explain and model a number of other phenomena, those pointed out in section 1, as well as covalent bonding, ferromagnetism, superconductivity, Bose–Einstein condensation, the formation of stars, the structure of hadrons, etc., always in accordance with experimental observations. Note also some recent attempts to do quantum information processing based solely on the use of the statistical properties of particles [35–37].

Despite all our understanding of particle statistics, it is still quite mysterious why (or how) fermions with common values in their internal degrees of freedom will resist being brought close together, as in the dramatic example of the formation of neutron stars, this resistance resulting in an effective force, completely different from the other interactions we know. This force is sometimes referred to as *Pauli pressure*. Equally intriguing is the tendency of photons to bunch together, as in the *Hanbury Brown and Twiss effect* [38], in which case the interaction seems to have exactly the opposite effect.

One interpretation for these effects lies in interference. This can best be observed using a beam splitter. Here we use the term *beam splitter* in a generic sense, referring not only to the common optical element (a partially silvered mirror) used with photons, but also to any device presenting an analogue behaviour for other kinds of particles, as was recently proposed for electrons, using a quantum dot and nanowires to direct the incoming and outgoing particles [39]. Note that two photons with the same frequency and polarization impinging simultaneously in a 50/50 beam splitter will always *bunch* [40], i.e. they will always come out together in the same output arm, as is shown in figure 3. Equivalently, two electrons with the same spin projection impinging simultaneously in a 50/50 beam splitter will always *antibunch* [39], i.e. they will always come out separately, one in each output arm, as illustrated in figure 4. When the particles meet in the beam splitter under the circumstances described above they are fully indistinguishable and the Symmetrization Postulate must then be taken into account. The bunching and antibunching of these particles are effects of particle statistics. Here, they can be interpreted as the result of constructive and destructive interference. Note that these results can easily be generalized: in such balanced beam splitters two indistinguishable particles will always bunch if they are bosons, and always antibunch if they are fermions. Another very interesting aspect is that performing simple path measurements on the output particles can offer us (probabilistic) information about their internal states.

Particle statistics, and in particular the Symmetrization Postulate, represent a limitation on Quantum Mechanics. There is nothing, apart from our empirical knowledge, that would lead us to impose such constraints. From the theoretical point of view, it is perfectly possible to consider

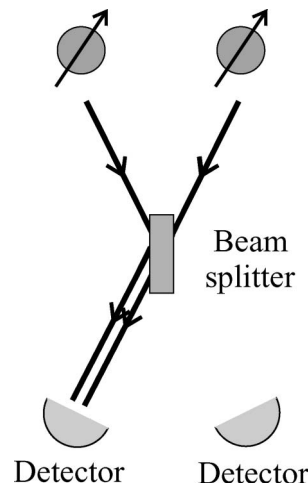


Figure 3. This figure represents two identical bosons bunching to the left out of a 50/50 beam splitter. If the two particles are initially in the same internal state—e.g. two photons with the same polarization, as represented by the arrows—then the Symmetrization Postulate imposes that they always leave the beam splitter in the same arm, either to the left or to the right, each possibility being equally likely. This phenomenon has been observed experimentally for photons [40].

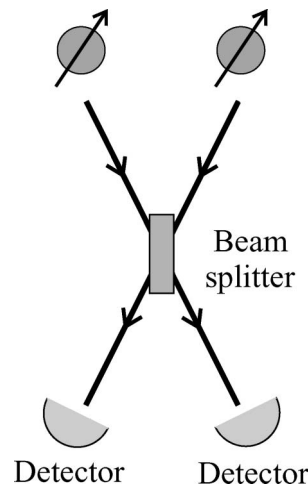


Figure 4. This figure represents two identical fermions antibunching out of a 50/50 beam splitter. If the two particles are initially in the same internal state—e.g. two electrons with aligned spins, as represented by the arrows—then the Symmetrization Postulate (or just Pauli’s Exclusion Principle) imposes that they always leave the beam splitter in separate arms. This phenomenon has recently been observed experimentally for electrons [39].

different constraints, i.e. to have quantum mechanical theories following *alternative statistics*, more general than the one we know. Moreover, it is also possible that such

alternative constraints haven't been masked by experiments performed so far.

Alternative statistics have actually been studied. For instance, *parastatistics* [41] proposes the following generalization of fermions and bosons:

- (a) *parafermions* of order  $n$ , for which a maximum of  $n$  indistinguishable such particles can be in a symmetric state, i.e. at most  $n$  particles can occupy the same state;
- (b) *parabosons* of order  $n$ , for which a maximum of  $n$  indistinguishable such particles can be in an antisymmetric state.

Note that the case  $n=1$  corresponds to the usual particle statistics and that, for all known particles, parastatistics is clearly violated for any larger  $n$ . Yet, in the early sixties, when the quark model was first proposed, it seemed that baryons (particles composed of three quarks, such as the proton) could only be described by a symmetric wave function, even though they were fermions. At the time it was considered that quarks could obey parastatistics [42], but the problem was later solved with the introduction of the *colour* internal degree of freedom in quarks [43]—the charge of the strong interaction—which implied an antisymmetric wave function to describe baryons, as one would expect according to the Symmetrization Postulate, and thus confirming particle statistics in the end.

Another alternative to particle statistics are *quons* [44], a continuous interpolation between fermions and bosons. In such a theory, the field operators describing the states of the particles obey the following relation:

$$\hat{a}_i \hat{a}_j^\dagger - q \hat{a}_j^\dagger \hat{a}_i = \delta_{ij} \hat{I}, \quad (25)$$

where  $q \in [-1, 1]$ . The limits  $q = -1$  and  $q = 1$  correspond to fermions and bosons respectively, where we recover the anticommutation and commutation relations given by equations (20) and (24). Quons offer us a setting to extend these relations infinitesimally (and continuously) and thus model and investigate small deviations to Fermi–Dirac and Bose–Einstein statistics. No such violations have been found up to this date. In fact, several high precision experiments have been performed recently looking for direct evidence of the violation of particle statistics or the spin-statistics connection and have not only confirmed that they hold—as one would expect—but also established very low probability bounds for a violation to occur [45–50].

Finally, let us mention *anyons*, quasiparticles in two dimensions that obey fractional statistics, i.e. the permutation of two of them can give *any* phase [51], not only the usual  $+1$  and  $-1$  associated to bosons and fermions. For example, the set composed of a charged particle orbiting around a magnetic flux tube has this property. Anyons are

believed to play a role in the fractional quantum Hall effect [52], and possibly also in high temperature superconductivity [53]. Recently, it was proposed to use anyons for quantum computation [54], exploring the intrinsic fault-tolerant properties that these objects can offer.

## 5. Summary and concluding remarks

Quantum Mechanics, a theory at the basis of current fundamental Physics, gives us an incredibly accurate description of Nature at the atomic scale. Yet, its descriptive power would be very limited without the introduction of the Symmetrization Postulate. This is a very simple rule that encodes in an elegant way the observed fact that all known quantum particles belong to one of two possible classes given their behaviour in indistinguishable situations: they are either *fermions* or *bosons*. We have no clear understanding of the origin of this behaviour, not even within the framework of a more sophisticated theory, and are thus happy (or forced) to accept it as an empirical rule. Yet, the Symmetrization Postulate is often neglected (if not ignored) when compared to the other postulates of the theory, even if it is not less mysterious than these, nor less fundamental as far as we know today. This unfair attitude has hindered the research on the origin of quantum statistics, as well as on the understanding of the origin of the Spin-Statistics connection, still today one of the greatest mysteries of theoretical Physics.

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## Appendix A: Second quantization formalism

The formalism used throughout most of this article, sometimes referred to as *first quantization*, is standard to describe non-relativistic Quantum Mechanics, but the *second quantization* formalism, introduced in this appendix, could have been used just as well. The second-quantized theory is more general though: it contains non-relativistic Quantum Mechanics, but it is a full-fledged relativistic theory that can describe new processes, such as the creation and annihilation of particle–antiparticle pairs.

The second quantization formalism is used in Quantum Field Theory, where different types of particles are described as quanta of different fields. For instance, a photon is a quantum of the electromagnetic field. Different photons, i.e. photons with different energies and polarization, correspond to different modes of the field. To each of these modes, i.e. to each possible state of the photons, we associate a *number state* that keeps track of the number of particles/quanta in that state/mode. Imagine the following restricted situation, where we have only:

- 2 photons with energy  $E_1$  and polarization  $H$  (horizontal);
- 0 photons with energy  $E_1$  and polarization  $V$  (vertical);
- 3 photons with energy  $E_2$  and polarization  $H$ ;
- 2 photons with energy  $E_2$  and polarization  $V$ .

Then, we could describe it by the following number state:

$$|2, 0, 3, 2\rangle = |2\rangle_{E_1H}, |0\rangle_{E_1V}, |3\rangle_{E_2H}, |2\rangle_{E_2V}. \quad (\text{A1})$$

Formally, we use the creation operator  $\hat{a}_m^\dagger$  and the *annihilation operator*  $\hat{a}_m$  to respectively increase or decrease by one the *population* of mode  $m$ . Note that  $\hat{a}_m^\dagger|0\rangle$  can also be used as a description of  $|1\rangle_m$ , with  $|0\rangle$  representing the *vacuum state*, where all modes are unpopulated. The population of a mode is given by the eigenvalue of the *number operator*:

$$\hat{N}_m \equiv \hat{a}_m^\dagger \hat{a}_m. \quad (\text{A2})$$

Note also that:

$$\hat{a}_m|0\rangle = 0. \quad (\text{A3})$$

This formalism can be applied not only to photons, but to bosons in general. In fact, it can also be applied to

fermions, although in this case the creation and annihilation operators will be different, as we can see in sections 3.1 and 3.2.

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