

Topologically protected quantization of work

Bruno Mera*

*Instituto de Telecomunicações, 1049-001 Lisboa, Portugal and
CeFEMA, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal*

Krzysztof Sacha

*Instytut Fizyki imienia Mariana Smoluchowskiego, Uniwersytet Jagielloński,
ulica Profesora Stanisława Lojasiewicza 11, PL-30-348 Kraków, Poland*

Yasser Omar

*Instituto de Telecomunicações, 1049-001 Lisboa, Portugal and
Departamento de Matemática, Instituto Superior Técnico,
Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal*

(Dated: February 15, 2019)

The transport of a particle in the presence of a potential that changes periodically in space and in time can be characterized by the amount of work needed to shift a particle by a single spatial period of the potential. In general, this amount of work, when averaged over a single temporal period of the potential, can take any value in a continuous fashion. Here we present a topological effect inducing the quantization of the average work. We find that this work is equal to the first Chern number calculated in a unit cell of a space-time lattice. Hence, this quantization of the average work is topologically protected. We illustrate this phenomenon with the example of an atom whose center of mass motion is coupled to its internal degrees of freedom by electromagnetic waves.

Topological phases of matter constitute a new paradigm in condensed matter physics. Remarkable examples are the Haldane anomalous insulator [1], an instance of the more general Chern insulators, and, even more generally, topological insulators and superconductors [2, 3]. The latter are symmetry protected topological phases of free fermions. Unlike the conventional phases of matter described by the Landau-Ginzburg theory in terms of a local order parameter [4], instead, topological insulators and superconductors are described by topological invariants, such as the holonomy of a flat connection, like the Zak phase [5, 6], or the Chern number of a vector bundle [7–10] over a 2-torus or an arbitrary Riemann surface, like the Thouless-Kohmoto-Nightingale-den Nijs (TKNN) invariant [11]. These topological invariants measure the non-trivial “twisting” of the wave-functions of the bulk, which are usually subject to certain generic symmetries like time-reversal, particle-hole or chiral symmetries. Topological insulators and superconductors were systematically classified, using K-theory [12], by Kitaev [13]; and using homotopy groups and Anderson localization, by Schnyder, Ryu, Furusaki and Ludwig [14, 15]. The resulting classification exhibits Bott-periodicity, 2-fold for the complex case and 8-fold for the real case, and is known as the periodic table of topological insulators and superconductors. Moreover, the bulk-to-boundary principle predicts that, when terminating the system to the vacuum, there will appear gapless modes living in the boundary of the system. These modes are topologically protected. One can understand the existence of these gapless modes by anomaly inflow arguments [16, 17].

Topological insulators and superconductors [2, 3, 18] are very attractive from the experimental point of view due to their robustness to perturbations and also due to the many potential applications to photonics, spintronics, quantum computing, and, more generally, to the emergent field of quantum technologies [19, 20].

Experimentally, one can study topological insulators and superconductors in quantum simulators which are versatile systems that can mimic behavior of other systems difficult to control in the laboratory. Among the physical platforms for the quantum simulation of topological matter, ultracold atoms in optical lattices [21, 22] and topological photonics [23] offer the most promising realizations. Quantum simulators have allowed for the realization of the topological insulators in one-dimensional (1D) [24–29], 2D [30–32] and even 4D space [33, 34] – the latter with the help of a synthetic dimension.

Thirty years ago Thouless proposed the idea of a topological charge pump where transport of charge, described by an adiabatically and periodically evolving Hamiltonian, was quantized and determined by the first Chern number calculated in the time-momentum space [35]. More concretely, if one has a one-dimensional translation invariant gapped system of free fermions on a lattice then, by adiabatically and periodically driving the system, the center of mass position is shifted, in one period of driving, by an integer multiple of the lattice constant. This integer is the first Chern number of the vector bundle of occupied states in the instantaneous ground states of the system, defined over the space of Bloch momenta and time – topologically, a torus. Direct observation of the Thouless quantum pump was demonstrated in a quan-

tum simulator where bosonic ultra-cold atoms were prepared in the Mott insulator phase in an optical lattice whose tunneling amplitudes were periodically modulated in time [36, 37].

In the present letter we consider a different phenomenon. Namely, we consider an atom constrained to move in 1D, with internal degrees of freedom subject to a space-time periodic potential coupling the internal states. In this case, it is possible, by preparing the system in a dressed state, that the atom experiences an effective synthetic electric field whose average work, in a period of driving and one wavelength, is quantized in units of the Planck constant \hbar . The quantization is topological in nature and robust against deformations of the system preserving the gap. In the following, we provide an explicit situation where this topological effect occurs and propose a way to experimentally realize it. The differences between this phenomenon and that of Thouless pumping are pointed out in Table I.

Let us consider an atom where the ground state energy level is characterized by the total angular momentum $F = 1$ and in the presence of an external magnetic field the energies E_{m_F} of the magnetic sublevels are split, $\Delta E = E_1 - E_0 = E_0 - E_{-1}$. If an atom is subjected to two counter-propagating circularly polarized electromagnetic waves of the frequency ω , the internal degrees of freedom of an atom and the electromagnetic fields can be described by the dressed-atom Hamiltonian which, within the rotating wave approximation, reads [21],

$$M(t, x) = \delta(t) (|1\rangle\langle 1| - |2\rangle\langle 2|) + \Omega(t, x)|0\rangle\langle 1| + \Omega^*(t, x)|0\rangle\langle 2| + \text{H.c.}, \quad (1)$$

where we assume that the detuning is oscillating in time due to the periodic modulation of the magnetic field, $\delta(t) = \Delta E(t) - \hbar\omega = [\gamma + \nu \cos(\tilde{\omega}t)]/2$, with the frequency $\tilde{\omega} \ll \omega$ and $\nu, \gamma \in \mathbb{R}$. The Rabi frequency depends periodically on time and space, $\Omega(t, x) = \alpha_1(t)e^{ikx} + \alpha_2(t)e^{-ikx}$ where k denotes the wave number of the electromagnetic waves while $\alpha_1(t) = \alpha \cos(\tilde{\omega}t - \pi/4)$ and $\alpha_2(t) = \alpha \cos(\tilde{\omega}t + \pi/4)$ describe periodic modulations of the amplitudes of the waves, with the same frequency as the frequency of the magnetic field modulation, where α is proportional to the dipole matrix element. The Hamiltonian $M(t, x)$ is periodic both in space and in time with the periods $\lambda = 2\pi/k$ and $T = 2\pi/\tilde{\omega}$, respectively, and can be written in a more compact form, $M(t, x) = \sum_{\mu=1}^3 B^\mu(t, x)J_\mu$ where $J_3 = |2\rangle\langle 2| - |0\rangle\langle 0|$, $J_1 - iJ_2 = \sqrt{2}(|0\rangle\langle 1| + |2\rangle\langle 0|)$ and

$$\begin{aligned} B^1(t, x) &= \alpha \cos(kx) \cos(\tilde{\omega}t), \\ B^2(t, x) &= \alpha \sin(kx) \sin(\tilde{\omega}t), \\ B^3(t, x) &= \gamma + \nu \cos(\tilde{\omega}t). \end{aligned} \quad (2)$$

When the atomic center of mass motion is coupled to its internal degrees of freedom certain geometric gauge fields arise [21, 38–41]. For simplicity, let us consider that

the atomic motion is restricted to one spatial dimension. The full Hamiltonian of the system is given by

$$H = \frac{p^2}{2m} + M(t, x), \quad (3)$$

where x and p are the atomic center of mass coordinate and momentum, m is the mass. We can solve the eigenvalue problem for $M(t, x)$, yielding the eigenvalues $\varepsilon_1(t, x) = |\mathbf{B}(t, x)| = \sqrt{\sum_\mu (B^\mu)^2}$, $\varepsilon_2(t, x) = 0$ and $\varepsilon_3(t, x) = -\varepsilon_1(t, x)$ and the corresponding eigenstates (dressed states of an atom) $|\eta_i(t, x)\rangle$. The most general solution of the Schrödinger equation will be given by a linear combination $\psi(t, x) = \sum_{i=1}^3 \Psi^i(t, x)|\eta_i(t, x)\rangle$. Writing the vector $\Psi = (\Psi^1, \Psi^2, \Psi^3)^T$, we get the time-dependent Schrödinger equation corresponding to the Hamiltonian (3) in the form

$$\left[i\hbar \left(\frac{\partial}{\partial t} + \mathcal{A}_0 \right) + \frac{\hbar^2}{2m} \left(\frac{\partial}{\partial x} + \mathcal{A}_1 \right)^2 - V - \mathcal{E} \right] \Psi = 0, \quad (4)$$

where $\mathcal{E} = \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and the matrices $\mathcal{A}_0 = (\langle \eta_i | \partial_t | \eta_j \rangle)$ and $\mathcal{A}_1 = (\langle \eta_i | \partial_x | \eta_j \rangle)$ are the components of the matrix-valued one-form $\mathcal{A} = (\langle \eta_i | d | \eta_j \rangle)$.

For the configuration of the electromagnetic waves and the detuning we have chosen, the eigenvalues $\varepsilon_i(t, x)$ are separated from each other by gaps for each (t, x) . If we prepare an atom in, e.g., the positive energy dressed state,

$$|\eta_1(t, x)\rangle = \frac{|2\rangle + z|0\rangle + z^2|1\rangle}{1 + |z|^2}, \quad z(t, x) = \frac{B^1 + iB^2}{|\mathbf{B}| + B^3}, \quad (5)$$

it will follow this state in the time evolution provided its kinetic energy is much smaller than the gap between the adjacent dressed state levels. Then, one can perform an adiabatic Born-Oppenheimer approximation and project the dynamics onto this state only [21, 39–41]. The resulting effective Schrödinger equation for the center of mass wave-function, $\phi(t, x) = \Psi^1(t, x)$, is that of a particle in the presence of an external gauge field, $qA_0/\hbar = i\langle \eta_1 | \partial_t | \eta_1 \rangle$ and $qA_1/\hbar = i\langle \eta_1 | \partial_x | \eta_1 \rangle$, and an effective scalar potential V_{eff} ,

$$i\hbar \frac{\partial \phi}{\partial t} = \left[\frac{1}{2m} (p - qA_1)^2 - qA_0 + V_{\text{eff}} \right] \phi, \quad (6)$$

with

$$V_{\text{eff}} = \frac{\hbar^2}{2m} g_{11}(t, x) + \varepsilon_1(t, x), \quad (7)$$

where $g_{11} = \sum_{j>1} |\langle \eta_1 | (\partial M / \partial x) | \eta_j \rangle|^2 / (\varepsilon_j - \varepsilon_1)^2$ is the 11th component of the *quantum metric* [41–43]. The charge q at this moment is arbitrary and was introduced for dimensional reasons.

Since the atom is constrained to move in a single space dimension, the only relevant component of the field strength tensor is the *synthetic electric field*

$$F_{01}(t, x) \equiv E(t, x) = -\frac{\hbar}{q} \frac{\mathbf{B} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} \times \frac{\partial \mathbf{B}}{\partial x} \right)}{|\mathbf{B}|^3}. \quad (8)$$

Because $\mathbf{B} \equiv (B^1, B^2, B^3)$ is space-time periodic, one can define a Chern number associated to the positive energy dressed state which will be minus twice the winding number of the map $(t, x) \mapsto \mathbf{B}(t, x)/|\mathbf{B}(t, x)| \in S^2$, where S^2 denotes the unit sphere in \mathbb{R}^3 . In particular, with $\alpha/\nu = 1$, using Eq. (2), we get a nontrivial Chern number -4 for $-1 < \gamma/\nu < 0$ and 4 for $0 < \gamma/\nu < 1$ and trivial elsewhere.

Similar result holds for the negative energy dressed state but with the Chern number being the opposite. The zero energy dressed state always has trivial Chern number.

The quantization of the Chern number amounts to having, on the unit cell of the space-time lattice, a quantized value for the flux $\int F_{01} dt dx = \int E(t, x) dt dx$ in units of $2\pi\hbar/q \equiv h/q$. Now $E(t, x) dx$ is, dimensionally, the amount of work, of the *electric field* per *charge*, under the displacement dx . The space-time lattice involved is simply $\Lambda = \{(t, x) = (mT, n\lambda), m, n \in \mathbb{Z}\}$. The interpretation of the quantized value of the Chern number is the following: the average over a period T of the work performed by the *electric field* E in the transport of a classical particle by a distance of a single space cell, i.e. $x \rightarrow x + \lambda$, is quantized in units of Planck's constant h .

A possible way to experimentally observe quantization of the average work, in the example we consider, can be done indirectly as follows. Take the time interval $[0, T]$ and consider a number N of instants t_i , $i = 1, \dots, N$. For each instant t_i , prepare an atom in the dressed state band with energy $\varepsilon_1(t_i, x)$ and described, at (t_i, x) , by $|\eta_1(t_i, x)\rangle$. We want the state of the center of mass degree of freedom of an atom to be strongly localized in a certain point $x(t_i)$ (i.e. much better than the size of a single space cell which is not a problem if the experiment is performed in the RF range where the wavelength λ is of the order of the meter), so the dynamics for time $t \in [t_i, T]$ is well described by the classical equations of motion:

$$m \frac{d^2 x}{dt^2}(t) = qE(t, x(t)) - \frac{\partial \varepsilon_1}{\partial x}(t, x(t)) - \frac{\hbar^2}{2m} \frac{\partial g_{11}}{\partial x}(t, x(t)). \quad (9)$$

We then measure the position of the atoms. in the period $[0, T]$. With the resulting trajectories $x_i(t)$, $t \in [t_i, T]$, $i = 1, \dots, N$, we can then differentiate with respect to time twice obtaining the acceleration. With this procedure we will get a profile of the total force field in the unit cell of Λ which we can compare to the theoretical predictions. Due to the localization of the center of mass of an atom, the observed profile should be the same and quantization of the time average of work of $E(t, x)$ can be confirmed. In Fig. 1 we show how the sampling of the accelerations of trajectories allows us to have the force profile on the unit cell. Additionally, we show the profile of $E(t, x)$ on the unit cell. The total force and the profile of $E(t, x)$ are

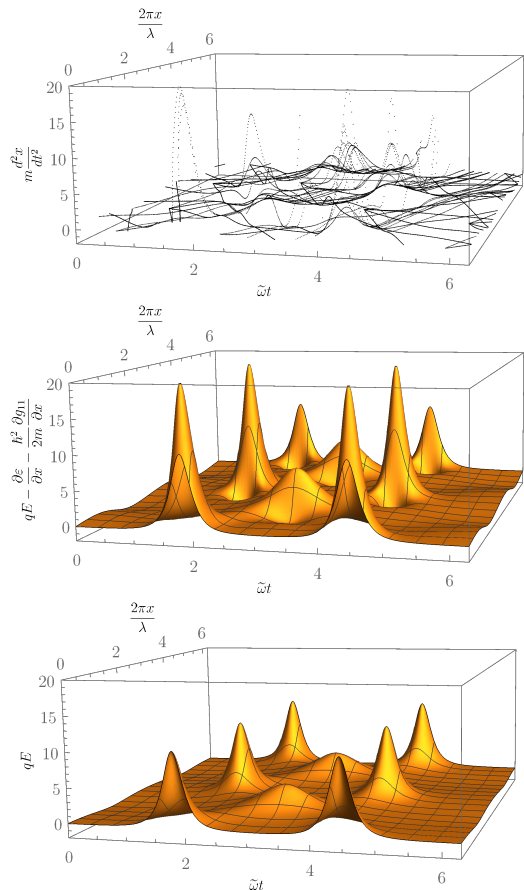


FIG. 1. Proposal for experimental demonstration of the quantization of the average work performed by the *electric field* $E(t, x)$. Top panel shows acceleration profile in a space-time unit cell obtained by integration of the classical equations of motion (9). For each initial condition, the resulting classical trajectory allows one to calculate the acceleration by differentiation of the trajectory with respect to time twice. In the middle panel we show the same profile obtained by plotting the total force acting on an atom. In the bottom panel the contribution from the *electric field* $E(t, x)$ is plotted only. The integration of $E(t, x)$ over a single space-time cell results in a quantized value which can be also obtained when one estimates the integral with the help of the points presented in the top panel. In the latter case, one has to first subtract the other contributions to the total force which are known from the theoretical description, cf. (9). Here $\mu = 0.5$ and $m = 1$. The Chern number is 4. Moreover, we take $q = \hbar = 1$.

qualitatively similar. The reason is that all contributions to the total force increase with the decrease of the gap function $\varepsilon_1(t, x)$.

We would like to remark that the phenomenon we describe is at the boundary between classical and quantum physics. Classical, since we want the states of the atoms to be strongly localized, so that the dynamics is classical. This is achieved by staying in the cold atom regime and not in the ultracold one. Quantum, since the atom will experience the effect of a *synthetic force* whose aver-

age work on the unit cell is quantized due to the quantum nature of the wave-function of the internal degrees of freedom of the atom. This is achieved by having a gap which is larger than the kinetic energy of the atom.

Finally, we would like to stress that the topological effect of work quantization considered in this letter and that of Thouless pumping are physically different, although mathematically similar, cf. Table I. Explicitly, in our case, it is the topology of quantum states over space-time and not of quantum states over Bloch momentum space that is involved. This topology is then reflected in the *quantization* of the average work of the synthetic electric field and not the quantization of the shift of the centre of mass of the system.

In summary, we have presented an effect in which the transport of a particle in the presence of a space-time periodic potential is characterised by a quantized av-

erage, over a period of the potential, amount of work needed to shift a particle by a single spatial period of the potential. The quantization was understood in terms of the topological twist of the vector bundle of dressed states. Moreover, we have provided an experimental procedure to probe this phenomenon.

We are grateful to Tomasz Kawalec for a fruitful discussion concerning experimental aspects. BM thanks the support from Fundação para a Ciência e a Tecnologia (Portugal), namely through programme POCH and projects UID/EEA/50008/2013 and IT/QuNet, as well as from the JTF project NQuN (ID 60478). The authors acknowledge the support from the project TheBlinQC supported by the EU H2020 QuantERA ERA-NET Cofund in Quantum Technologies and by FCT (QuantERA/0001/2017)

<i>Property</i>	<i>Top. effect</i>	<i>Work quantization</i>	<i>Thouless pumping</i>
Parameter space topologically a torus T^2		Space-time	Bloch momenta and time
Gapped Hamiltonian		$M(t, x) = \sum_{\mu=1}^3 B^{\mu}(t, x) J_{\mu}$	$H(t, k) = \sum_{\mu=1}^3 d^{\mu}(t, k) \sigma_{\mu}$ (in the simplest scenario of a two-band system)
Wave-functions (sections)		Dressed states $ \eta(t, x)\rangle$	Bloch states $ u(t, k)\rangle$
Gauge field		$-iqA(t, x)/\hbar = \langle \eta(t, x) d \eta(t, x) \rangle$ acts as an effective external field	$A(t, k) = \langle u(t, k) d u(t, k) \rangle$ manifests through coupling to an external field
Field strength		$F(t, x) = dA$	$F(t, k) = dA$
1st Chern number		Average work performed by $E(t, x)$ on the unit cell of space-time lattice	Shift of the center of mass of a system $x_{cm} = \langle x \rangle$ in one period of driving

TABLE I. Comparison between charge pumping and quantization of work. The σ_{μ} 's denote the usual Pauli matrices.

* bruno.mera@tecnico.ulisboa.pt

[1] F. D. M. Haldane, "Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization

of the "Parity Anomaly"," *Phys. Rev. Lett.* **61**, 2015–2018 (1988).

[2] B Andrei Bernevig and Taylor L Hughes, *Topological insulators and topological superconductors* (Princeton university press, 2013).

[3] M. Z. Hasan and C. L. Kane, "Colloquium: Topological

- insulators,” *Rev. Mod. Phys.* **82**, 3045–3067 (2010).
- [4] PW Anderson, *Basic Notions of Condensed Matter Physics* (Westview Press, Boulder, CO, 1997).
- [5] J. Zak, “Berry’s phase for energy bands in solids,” *Phys. Rev. Lett.* **62**, 2747–2750 (1989).
- [6] M. V. Berry, “Quantal phase factors accompanying adiabatic changes,” *Proc. R. Soc. Lond. A* **392**, 45–57 (1984).
- [7] Mikio Nakahara, *Geometry, topology and physics* (CRC Press, 2003).
- [8] Shigeyuki Morita, *Geometry of differential forms*, Vol. 201 (American Mathematical Soc., 2001).
- [9] Norman Earl Steenrod, *The topology of fibre bundles*, Vol. 14 (Princeton University Press, 1999).
- [10] Raoul Bott and Loring W Tu, *Differential forms in algebraic topology*, Vol. 82 (Springer Science & Business Media, 2013).
- [11] D. J. Thouless, Mahito Kohmoto, M. P. Nightingale, and M. den Nijs, “Quantized Hall conductance in a two-dimensional periodic potential,” *Physical Review Letters* **49**, 405 (1982).
- [12] Max Karoubi, *K-theory: An introduction*, Vol. 226 (Springer Science & Business Media, 2008).
- [13] Alexei Kitaev, “Periodic table for topological insulators and superconductors,” in *AIP Conference Proceedings*, Vol. 1134 (AIP, 2009) pp. 22–30.
- [14] Andreas P. Schnyder, Shinsei Ryu, Akira Furusaki, and Andreas W. W. Ludwig, “Classification of topological insulators and superconductors in three spatial dimensions,” *Phys. Rev. B* **78**, 195125 (2008).
- [15] Andreas P. Schnyder, Shinsei Ryu, Akira Furusaki, and Andreas W. W. Ludwig, “Classification of topological insulators and superconductors,” in *AIP Conference Proceedings*, Vol. 1134 (AIP, 2009) pp. 10–21.
- [16] Shinsei Ryu, Joel E. Moore, and Andreas W. W. Ludwig, “Electromagnetic and gravitational responses and anomalies in topological insulators and superconductors,” *Physical Review B* **85**, 045104 (2012).
- [17] Edward Witten, “Fermion path integrals and topological phases,” *Reviews of Modern Physics* **88**, 035001 (2016).
- [18] Xiao-Liang Qi and Shou-Cheng Zhang, “Topological insulators and superconductors,” *Rev. Mod. Phys.* **83**, 1057–1110 (2011).
- [19] Yang Xu, Ireneusz Miotkowski, Chang Liu, Jifa Tian, Hyungdo Nam, Nasser Alidoust, Jiuning Hu, Chih-Kang Shih, M Zahid Hasan, and Yong P Chen, “Observation of topological surface state quantum hall effect in an intrinsic three-dimensional topological insulator,” *Nature Physics* **10**, 956 (2014).
- [20] Stephen Ornes, “Core concept: Topological insulators promise computing advances, insights into matter itself,” *Proceedings of the National Academy of Sciences* **113**, 10223–10224 (2016).
- [21] Nathan Goldman, G Juzeliūnas, P Öhberg, and Ian B Spielman, “Light-induced gauge fields for ultracold atoms,” *Reports on Progress in Physics* **77**, 126401 (2014).
- [22] N Goldman, JC Budich, and P Zoller, “Topological quantum matter with ultracold gases in optical lattices,” *Nature Physics* **12**, 639 (2016).
- [23] Tomoki Ozawa, Hannah M Price, Alberto Amo, Nathan Goldman, Mohammad Hafezi, Ling Lu, Mikael Rechtsman, David Schuster, Jonathan Simon, Oded Zilberberg, *et al.*, “Topological photonics,” arXiv preprint arXiv:1802.04173 (2018).
- [24] Filippo Cardano, Alessio D’Errico, Alexandre Dauphin, Maria Maffei, Bruno Piccirillo, Corrado de Lisio, Giulio De Filippis, Vittorio Cataudella, Enrico Santamato, Lorenzo Marrucci, *et al.*, “Detection of zak phases and topological invariants in a chiral quantum walk of twisted photons,” *Nature communications* **8**, 15516 (2017).
- [25] Takuya Kitagawa, Matthew A Broome, Alessandro Fedrizzi, Mark S Rudner, Erez Berg, Ivan Kassal, Alán Aspuru-Guzik, Eugene Demler, and Andrew G White, “Observation of topologically protected bound states in photonic quantum walks,” *Nature communications* **3**, 882 (2012).
- [26] Eric J Meier, Fangzhao Alex An, and Bryce Gadway, “Observation of the topological soliton state in the su–schrieffer–heeger model,” *Nature communications* **7**, 13986 (2016).
- [27] Marcos Atala, Monika Aidelsburger, Julio T Barreiro, Dmitry Abanin, Takuya Kitagawa, Eugene Demler, and Immanuel Bloch, “Direct measurement of the zak phase in topological bloch bands,” *Nature Physics* **9**, 795 (2013).
- [28] P St-Jean, V Goblot, E Galopin, A Lemaître, T Ozawa, L Le Gratiet, I Sagnes, J Bloch, and A Amo, “Lasing in topological edge states of a one-dimensional lattice,” *Nature Photonics* **11**, 651 (2017).
- [29] Eric J. Meier, Fangzhao Alex An, Alexandre Dauphin, Maria Maffei, Pietro Massignan, Taylor L. Hughes, and Bryce Gadway, “Observation of the topological anderson insulator in disordered atomic wires,” *Science* **362**, 929–933 (2018), <http://science.sciencemag.org/content/362/6417/929.full.pdf>.
- [30] Monika Aidelsburger, Michael Lohse, C Schweizer, Marcos Atala, Julio T Barreiro, S Nascimbene, NR Cooper, Immanuel Bloch, and Nathan Goldman, “Measuring the chern number of hofstadter bands with ultracold bosonic atoms,” *Nature Physics* **11**, 162 (2015).
- [31] BK Stuhl, H-I Lu, LM Ayccock, D Genkina, and IB Spielman, “Visualizing edge states with an atomic bose gas in the quantum hall regime,” *Science* **349**, 1514–1518 (2015).
- [32] Matthias Tarnowski, F Nur Ünal, Nick Fläschner, Benno S Rem, André Eckardt, Klaus Sengstock, and Christof Weitenberg, “Characterizing topology by dynamics: Chern number from linking number,” arXiv preprint arXiv:1709.01046 (2017).
- [33] Michael Lohse, Christian Schweizer, Hannah M Price, Oded Zilberberg, and Immanuel Bloch, “Exploring 4d quantum hall physics with a 2d topological charge pump,” *Nature* **553**, 55 (2018).
- [34] Oded Zilberberg, Sheng Huang, Jonathan Guglielmon, Mohan Wang, Kevin P Chen, Yaacov E Kraus, and Mikael C Rechtsman, “Photonic topological boundary pumping as a probe of 4d quantum hall physics,” *Nature* **553**, 59 (2018).
- [35] J. Thouless, D, “Quantization of particle transport,” *Physical Review B* **27**, 6083 (1983).
- [36] Michael Lohse, Christian Schweizer, Oded Zilberberg, Monika Aidelsburger, and Immanuel Bloch, “A thouless quantum pump with ultracold bosonic atoms in an optical superlattice,” *Nature Physics* **12**, 350 (2016).
- [37] Nakajima Shuta, Tomita Takafumi, Taie Shintaro, Ichinose Tomohiro, Ozawa Hideki, Wang Lei, Troyer

- Matthias, and Takahashi Yoshiro, “Topological Thouless pumping of ultracold fermions,” *Nature Physics* **12**, 296 (2016).
- [38] G. Juzeliūnas and P. Öhberg, “Slow light in degenerate fermi gases,” *Phys. Rev. Lett.* **93**, 033602 (2004).
- [39] Jean Dalibard, Fabrice Gerbier, Gediminas Juzeliūnas, and Patrik Öhberg, “Colloquium: Artificial gauge potentials for neutral atoms,” *Rev. Mod. Phys.* **83**, 1523–1543 (2011).
- [40] J. Ruseckas, G. Juzeliūnas, P. Öhberg, and M. Fleischhauer, “Non-abelian gauge potentials for ultracold atoms with degenerate dark states,” *Phys. Rev. Lett.* **95**, 010404 (2005).
- [41] Dariusz Chruscinski and Andrzej Jamiolkowski, *Geometric phases in classical and quantum mechanics*, Vol. 36 (Springer Science & Business Media, 2012).
- [42] Raffaele Resta, “The insulating state of matter: a geometrical theory,” *The European Physical Journal B* **79**, 121–137 (2011).
- [43] Michael Kolodrubetz, Dries Sels, Pankaj Mehta, and Anatoli Polkovnikov, “Geometry and non-adiabatic response in quantum and classical systems,” *Physics Reports* **697**, 1–87 (2017).